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Beyond the Frame Rule: Static Inlining in Separation Logic

Master's Thesis

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Abstract

Various formal verification techniques can be used to automatically verify the absence of errors in programs. This provides an advantage over testing approaches, namely the guarantee that a program is correct for any possible execution. However, such approaches often require a user to provide additional specifications to guide the verification, in the form of loop invariants and method preconditions and postconditions, which places a burden on the user. When no specifications are provided, verifiers usually report potential errors which are not actual errors, hence lowering confidence in error reporting.

Users might want to learn quickly (without providing too many specifications) and with high confidence whether a program is incorrect. This would speed up the development and verification process by only having to provide specifications when one is fairly certain that the program is correct. Static inlining, that is inlining of method calls and unrolling of loop iterations, is an interesting approach to tackle this issue. Using approaches based on static inlining, verifiers could inform a user of the existence of fundamental errors, errors for which no annotation can make the program verify. The existence of fundamental errors indicates an error in the program itself, informing the user this program cannot be verified, without the user needing to waste time and energy in the search of the right annotation.

One would expect that errors reported in an inlined program always correspond to fundamental errors in the original program, since annotations only serve as approximations of method calls and loops. Surprisingly, this is not always the case. Indeed, Viper, a verification infrastructure for permission-based reasoning, partly based on separation logic and implicit dynamic frames with fractional permissions, has special features (such as permission introspection) which give rise to examples where this is not the case. These examples have all in common that they do not satisfy the frame rule.

In this thesis, we find (and prove correct) a soundness condition which

characterizes the set of Viper programs for which static inlining is sound, that is errors in the inlined program correspond to fundamental errors in the original program. We define a parametric language which generalizes Viper, where program states are elements of a separation algebra, define the soundness condition in terms of this language, and prove the soundness property of inlining under this soundness condition, using the proof assistant Isabelle/HOL. We then show how one can instantiate this parametric language to transfer the results to Viper. We also explore a completeness property of static inlining, and consider extensions to different loop semantics and different ways of inlining.

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Contents

Chapter 1

Introduction

1.1 Early detection of fundamental errors in Viper

1.1.1 Formal verification and the burden of annotating

Various formal verification techniques can be used to automatically verify the absence of errors in programs. This provides an advantage over testing approaches, namely the guarantee that a program is correct for any possible execution. In practice, the program is tested against a specification written by the programmer. A common way of expressing a specificiation is via a contract, containing:

1. A precondition, namely a specification describing under which conditions the program should be executed.

2. A postcondition, namely a specification describing a condition which has to hold after the execution of the program, when executed under the conditions described by the precondition.

As an example, imagine a program whose aim is to find a path between two cities, named A and B. A precondition for this program could be "A and B are the names of two existing cities", and a postcondition could be "if a path between A and B exists, then the output is a valid path between A and B, otherwise the output is an empty path". If this annotated program verifies, this means that every time it is called with two cities for which there exists a solution, then a solution is found and returned by this program.

Ideally, formal verification should be fully automated. That is, starting with a program annotated with a precondition and a postcondition, it should automatically check whether the contract is satisfied by the program. However, this problem is generally undecidable. One source of undecidability is the existence of statically unbounded loops.

One solution to overcome this theoretical limitation is to expect the programmer to annotate loops with loop invariants. A loop invariant is also a contract: If the loop invariant holds before the execution of an iteration of the loop, it has to hold afterwards. Loops in the program can then be approximated solely in terms of this contract. If the invariant holds before the loop, then it holds after the loop. Given a loop invariant, checking whether it is actually an invariant for a loop is decidable.[1] Moreover, checking whether the modified program (where loops are approximated by their contracts) satisfies its contract is also decidable.[2] The verification of a program with annotated loops therefore becomes a decidable problem, but not yet a practical one.

In practice, verifying such a program has a high computational complexity, and can become an intractable problem. Another useful improvement to reduce this complexity is to use modular verification. The idea is to divide the program in smaller methods, each of them doing a small part of the overall job. The programmer has to then annotate these methods with contracts, consisting of a precondition and a postcondition. Method calls are approximated by their contracts, which makes verification tractable. Finally, verification succeeds when each method satisfies its contract under the assumption that all other methods satisfy their contract. In this case, the whole program is verified.[3]

The required additional specifications described above, loop invariants and method preconditions and postconditions, place a burden on programmers who want to formally verify their programs. Indeed, on top of the initial effort of writing a program and an initial contract for this program, programmers also need to annotate the program with these additional specifications. However, if the annotations are not strong enough to prove the desired property, then the verifiers will report potential errors which are not actual errors, but spurious errors. As a result, the programmer needs to often spend a lot of time adjusting the annotations until they are strong enough for the verifier to prove the program correct, assuming the program is correct.

If the program is incorrect (and hence the verifier always reports that there is a potential error), then the programmer cannot easily distinguish whether the verifier reports an error because the annotation is not strong enough or whether the program is incorrect. Hence the progammer may waste a lot of time first trying to strengthen the annotations until he or she figures out that the program is actually incorrect and then adjusts the program. It would therefore be useful for programmers to learn quickly and with high

[1] More precisely, checking if an assertion is actually an invariant for a loop is decidable if the underlying theory is decidable.

[2] It is decidable if the underlying theory is decidable.

[3] This verification only guarantees partial correctness, namely the property that the program is correct when it terminates.

confidence whether a program is incorrect, before diving into the process of providing these specifications. This would speed up the development and verification process by only having to provide specifications when one is fairly certain that the program is correct.

1.1.2 Towards a useful feature: Static inlining to the rescue

Stratified inlining [12] is an approach which has the potential to improve this situation by iteratively performing static inlining of methods and loops, that is inlining of method calls and unrolling of loop iterations up to a bounded depth, in order to detect errors. This approach has been successfully applied to the intermediate verification language Boogie [2] in the tool Corral [13].

However, this approach based on static inlining cannot in general be straightforwardly applied to other intermediate verification languages. The intermediate verification language Viper [15] is an example where the use of static inlining to detect incorrect programs is unclear, mainly because of Viper's powerful support for fractional permissions to heap locations. Let us first illustrate the idea of static inlining with Viper, in a case where there no are no issues.

A fractional permission, namely a rational number between 0 and 1, is associated to a heap location (a pair of a reference x and a field f) in a program state. At least some (that is non-zero) permission is required for a program to read a heap location, and full permission (1) is required to write a heap location. Viper has also a special kind of assertion, accessibility predicates, in the form of `acc(x.f, p)`. This assertion represents a fractional permission of p to the heap location $x.f$. `acc(x.f)` is a special form of it, where p is 1, that is full permission to $x.f$. Take the Viper program shown on Listing 1.1 as an example.

The situation is the following: The programmer has finished writing a program, with an initial contract, basically saying "If the method *example* is called with full permission to *x.f*, then no error happens". Without any loop invariant, one of Viper's verifier would report an error message on line 12 (underlined with a dotted red line), saying that there is not enough permission held by the program state to read and modify the value of *x.f*. This issue can be solved by annotating the loop with the loop invariant commented on line 9, `acc(x.f)`, since there is enough permission to *x.f* held by the program state before the loop. Since it suffices to annotate the program to resolve this error, this is a not an actual error of the program, but a spurious error since the original annotation was too weak. In other words, this does not represent a "fundamental error". Let us define more precisely what we call a "fundamental error".

Listing 1.1: An example where an early error detection feature would be useful (original).

```
1  field f: Rational
2
3  method example(x: Ref, n: Int)
4      requires acc(x.f)
5      ensures true
6  {
7
8      var y: Ref
9      assume x != y
10     var i: Int := 0
11     while (i < n)
12         // invariant acc(x.f)
13         // invariant i >= 0
14     {
15         x.f := x.f + 1 / (i + 1)
16         x.f := x.f + 1 / (i - 1)
17         i := i + 1
18     }
19
20 }
```

Definition 1.1 *(Informal) A program contains a* ***fundamental error*** *if it is not possible to provide an annotation such that the program verifies.*

Annotating the loop with the invariant on line 9 makes the permission error on line 12 disappear. Another error is then reported on line 12, with the error message "divisor $i + 1$ might be zero". Indeed, since the division by 0 is not defined, dividing (potentially) by zero yields an error. As before, this does not correspond to a fundamental error: $i = 0$ during the first iteration, thus $i + 1 = 1 \neq 0$. Furthermore, i is incremented at each iteration, thus $i + 1$ will never be zero. Adding the invariant `i >= 0` commented on line 10 solves this issue.

The next error is reported on line 13 (underlined in red), another potential division by zero. However, this corresponds to a fundamental error. Indeed, $i - 1$ will be zero during the second iteration. This is therefore an example of an error we want to report. As shown on Listings 1.2, unrolling the loop enables the verifier to easily detect this fundamental error. Moreover, the verifier can report a more precise error message, by reporting which iteration contains the error. The usefulness of this feature is a main motivation for this thesis. However, as shown below, understanding when the inlined program can be used to find fundamental errors is non-trivial in Viper, as opposed to Boogie.

Listing 1.2: An example where an early error detection feature would be useful (inlined).

```
1  field f: Rational
2
3  method example(x: Ref, n: Int)
4      requires acc(x.f)
5      ensures true
6  {
7      var i: Int := 0
8      if (i < n)
9      {
10         x.f := x.f + 1 / (i + 1)
11         x.f := x.f + 1 / (i - 1)
12         i := i + 1
13         if (i < n)
14         {
15             x.f := x.f + 1 / (i + 1)
16             x.f := x.f + 1 / (i - 1)
17             i := i + 1
18             if (i < n)
19             {
20                 ...
21             }
22         }
23     }
24 }
```

1.1.3 Permission introspection in Viper: A fly in the ointment

The previous subsections explain why a feature to detect fundamental errors early, based on static inlining, would be useful in the practical context of formal verification. Therefore, one important part of this thesis is to explore the following *soundness* property of static inlining:

Definition 1.2 *(Informal) Static inlining is **sound** if errors in the inlined program correspond to fundamental errors in the original program.*

This thesis focuses partly on Viper for two reasons. The first reason is that Viper is an intermediate language with many front-ends built on top of it, as illustrated in Figure 1.1. Therefore, working directly at the Viper level enables to leverage this work for many different languages. The relationship between static inlining at the front-end level and at the Viper level is however not clear, and may need further exploration.

The second and more important reason is the existence of Viper programs for which static inlining is unsound (see Chapter 3 for examples). All Viper programs for which static inlining is unsound have a common point: They contain statements which break the frame rule (see Chapter 2). Moreover,

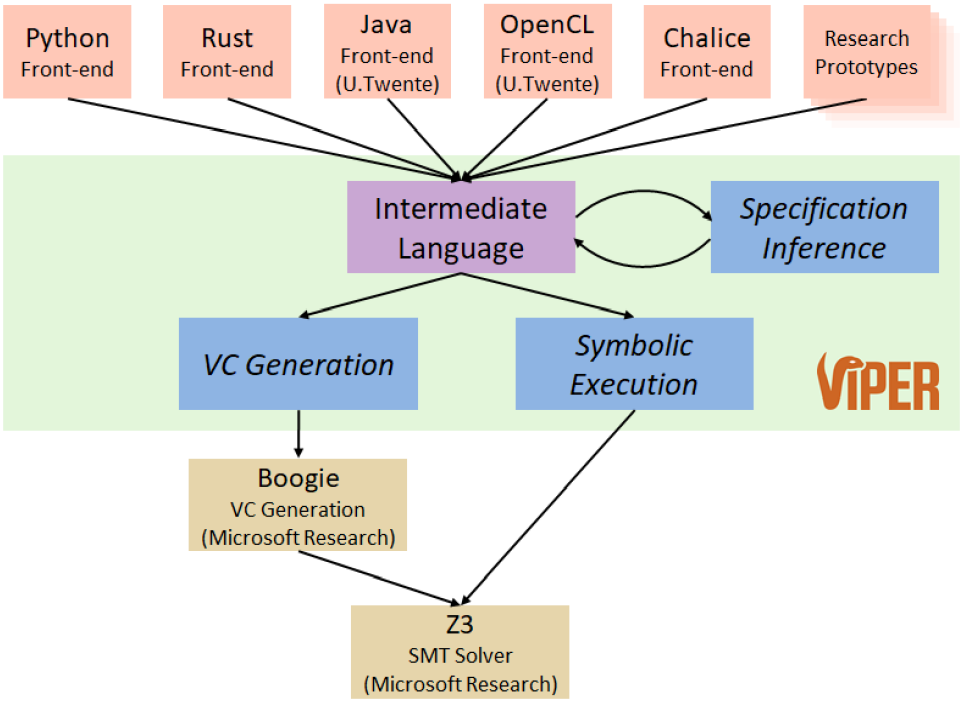


Figure 1.1: An illustration Viper architecture (figure taken from [15]).

static inlining is always sound for Viper programs where all statements respect the frame rule. This is probably a reason why this topic has not been explored before, since verification languages usually respect the frame rule.

The *frame rule* does not always hold in Viper, because of its advanced support for permissions, which encode ownership of parts of the heap. Amongst others, Viper has a built-in feature called **`perm`**, which evaluates to the permission amount held to a field location by the program state. For example, **`perm`**`(x.f)` returns the amount of permission currently held to *x.f*. Permission introspection is a powerful feature which allows, for example, to encode a leak check for monitors (see [15], Section 2.2), or to branch on the permission amount held in the state. It is a powerful debugging tool. It is also a reason why it is possible to write a Viper program for which static inlining is not sound. We show in Chapter 3 three examples illustrating how permission introspection makes static inlining unsound in Viper programs.

1.2 Problem statement: The meaning of inlining

The core of this thesis is the exploration of the *soundness* property of static inlining, in a general model which abstracts Viper. We illustrate some parts of the problem this thesis focuses on in Figure 1.2. We start with a program

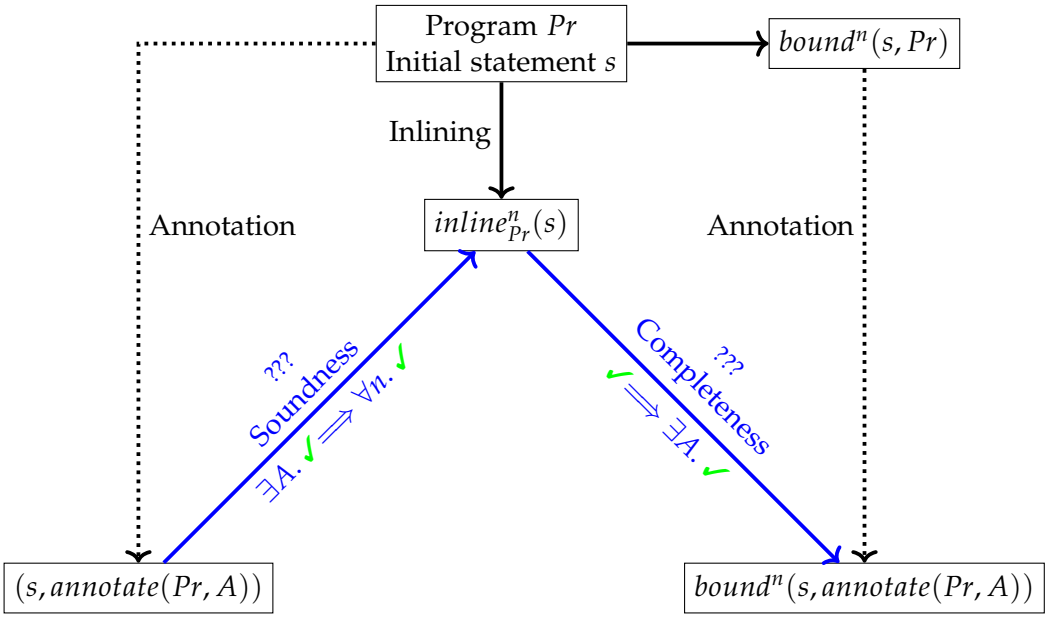


Figure 1.2: Illustration of the problem.

Pr, that is a set of methods, and an initial statement s. This initial statement s is the entry point of the program, similar to the body of a *main* method, and can call methods from the program. Our aim, formalized by the *soundness* property, is to use static inlining to answer the following question:

Is there any fundamental error in the program Pr, with the entry point s?

This question is represented on the left side of Figure 1.2. In particular, we represent on the diagram the program inlined up to the depth bound n as $inline^n_{Pr}(s)$. The definition of a fundamental error relates to annotated versions of the program, we therefore denote $annotate(Pr, A)$ the program Pr annotated with an annotation A. This annotation A represents all the annotations of the program, that is one precondition and one postcondition for each method, and one invariant for each loop.

Since there exist examples where *soundness* does not hold, the problem is actually about determining under which condition an error reported in the inlined program is a fundamental error. This condition is represented as "???" on the Figure, and the *soundness* property is represented by its contraposition: If there exists an annotation A such that the program verifies (in this case there are no fundamental errors), then the inlined program (up to any finite depth) should verify.

Moreover, we also want to know how efficient static inlining is in detecting fundamental errors. We call this property *completeness*. It is obvious that inlining up to a depth of n can only give us information on similarly bounded

executions of the original program. Therefore, we first define a bounded program $bound^n(s, Pr)$ which is a program behaving similarly to the original one, in which we simply stop the execution when we reach a depth greater than the bound. *Completeness* means that if no error is reported in the inlined program (the inlined program verifies), then there is no fundamental error in $bound^n(s, Pr)$ (the program can be annotated in a way which makes it verify).

On top of these two properties, *soundness* and *completeness*, we also present two extensions. The first is to explore different possibilities for defining the semantics of *while* loops in Viper, which then influence the *soundness* and *completeness* of static inlining. The second is to explore other useful ways of inlining (we only presented one way of inlining in this introduction, probably the simplest one).

Related work. A similar problem is tackled in [3], but with fundamental differences. In this work, the authors define a simple *while* language (with no method calls) and an operational semantics for this language. They mostly consider two verification workflows. A deductive verification workflow, which approximates loops with invariants, and a bounded model checking workflow, where loops are unrolled up to a fixed bound. They prove a soundness property and a completeness property for both workflows, with respect to the operational semantics. In this thesis, we do not consider any operational semantics, but study the direct link between the verification of the inlined program and the verification of the original annotated program. Moreover, the parametric language defined in this thesis is based on Viper, a concrete verification infrastructure with powerful features. One key difference is that we consider in this thesis features which make inlining unsound, and explore the conditions under which inlining is sound.

1.3 Overview of the thesis

This thesis is structured as follows. Chapter 2 presents some background on separation logic and Viper, needed to understand the rest. In Chapter 3, we define a desired soundness property for static inlining, and show examples of Viper programs for which this property does not hold. Based on these examples, we define two properties of statements, **mono** and **framing**, which are necessary to describe the conditions under which inlining is sound.

Since these properties and the soundness of inlining are more general than Viper, we define a general framework based on an abstraction of Viper, and prove soundness of static inlining in this framework. This is the core of this thesis. Chapter 4 defines a separation algebra [5] which abstracts the Viper state model. Building on this separation algebra, Chapter 5 defines a parametric language which abstracts the Viper language. Chapter 6 formalizes static

inlining, and proves soundness of static inlining in the general framework, under a formal condition based on the **mono** and **framing** properties. The definition of this framework with the proof of soundness of static inlining has been formalized in Isabelle/HOL [16] (see Appendix A).

Soundness is not the only interesting property about static inlining. Indeed, we are also interested in a completeness property of static inlining. Chapter 7 defines a completeness property for static inlining, and presents examples where this completeness property does not hold in Viper. We then sketch how to prove this completeness property in our general framework, under some assumptions.

Chapter 8 discusses how to instantiate the general framework for Viper, to leverage its theoretical results. We sketch a way to prove for Viper the assumption needed for the completeness property for inlining in the general framework. Moreover, we explain how to use this instantiation to create two useful features for Viper: An early error detection feature based on static inlining, and a feature to speed-up reverification[4] based on caching and the properties **mono** and **framing**.

Chapter 9 proposes and explores two extensions for this work. The first idea is to extend this work to different loop semantics, and the second idea is to explore different ways of inlining to gain more information.

Finally, Appendix A shows the Isabelle/HOL formalization of the general framework (separation algebra and parametric language) in which we prove soundness of static inlining, and an instantiation of some renaming functions necessary to express the semantics of method calls and for inlining method calls.

[4]That is, verification after small modifications of a first verified version.

Chapter 2

Background

2.1 Separation logic and fractional permissions

Separation logic [17, 21] is an extension of Hoare logic [10] for reasoning about programs which manipulate shared mutable data structures. It is based on the notion of partial heaps, and enables local reasoning thanks to the frame rule.

Partial heaps and states. A partial heap represents a part of the heap. A partial heap is formally a partial function, which maps heap locations (pairs of a reference and a field) to values. Two partial heaps are compatible if they are defined on disjoint domains, and their union is the partial heap which combines their mappings. A state consists of a partial heap and a store, which maps the names of local variables to their values. Two states are compatible if they agree on all values of local variables and their partial heaps can be combined.

Separating conjunction. Separation logic introduces the separation conjunction connective, written $*$. For two assertions A and B, a state φ satisfies the assertion $A * B$ if and only if there exist two states φ_1 and φ_2 such that φ is the combination of φ_1 and φ_2, φ_1 satisfies A, and φ_2 satisfies B.

Points-to assertions. An important type of assertion introduced by separation logic is the points-to assertion. As an example, $x.f \mapsto v$ (read $x.f$ points to v) where v is some value is satisfied by a state if and only if its partial heap maps the heap location $x.f$ to the value v.[1] In particular, combined with the separating conjunction, the assertion $x.f \mapsto v_1 * y.f \mapsto v_2$ is satisfied by a state if and only if its heap is defined at least for two heaps locations: $x.f$ and $y.f$, mapping $x.f$ to v_1 and $y.f$ to v_2. In particular, this assertion implies

[1] We describe here the intuitionistic version of separation logic, and not the classical one.

that $x.f$ and $y.f$ are disjoint parts of the heap. Therefore, $x.f \mapsto v$ represents ownership of the heap location $x.f$. In the logic, a state needs to own a heap location to read it or write it.

Hoare triples in separation logic. Separation logic slightly modifies the meaning of a Hoare triple. A Hoare triple is written $\{P\}\ C\ \{Q\}$, where C is a program, and P and Q are assertions (P is a precondition and Q a postcondition). In separation logic, $\{P\}\ C\ \{Q\}$ means that if C executes from a state satisfying the precondition P, then nothing will go wrong and the final state will satisfy the postcondition Q. In particular, the precondition P expresses which heap locations can be accessed and modified by the program C.

Local reasoning: The frame rule. On top of the standard rules from Hoare logic, separation logic introduces a new important rule, the frame rule, which enables local reasoning. The frame rule is defined as follows:

$$\frac{\{P\}\ C\ \{Q\}}{\{P * R\}\ C\ \{Q * R\}}\ \mathsf{mod}(C) \cap \mathsf{fv}(R) = \emptyset$$

$mod(C)$ is the set of local variables modified by C, and $fv(R)$ is the set of free variables in the assertion R. This rule means in essence that if a program executes correctly in a small state satisfying P, then it will also execute correctly in a bigger state satisfying $P * R$, and its execution will not affect the additional part of the state R. R should not interfere with C, that is R should not say anything about variables which are modified by C.

Fractional permissions. Fractional permissions [4] are a generalization of the points-to assertion. A points-to assertion is annotated with a fractional permission $q \in \mathbb{Q} \cap (0, 1]$, noted $x.f \overset{q}{\mapsto} v$, and partial heaps map heap locations to pairs of a permission and a value. Having some fractional permission to a location enables a state to read the value of this location. However, a state can write a heap location if and only if it has a permission of 1 to this location (full permission). Partial heaps with permissions can be combined by adding the permissions they hold, if they agree on the values of heap locations they both define, and if the sum of permissions is at most 1. Moreover, assertions can now be combined as follows, for permission amounts q_1 and q_2:

$$x.f \overset{q_1}{\mapsto} v_1 * x.f \overset{q_2}{\mapsto} v_2 \iff \begin{cases} x.f \overset{q_1+q_2}{\mapsto} v_1 & \text{if } v_1 = v_2 \wedge q_1 + q_2 \leq 1 \\ \bot & \text{otherwise} \end{cases}$$

Fractional permissions are especially useful in concurrent separation logic [19], where they allow concurrent reading of the same heap location.

2.2 Viper

Viper [15] is a powerful verification infrastructure with an intermediate verification language, based on implicit dynamic frames [14, 24, 20] (which are based on separation logic) and fractional permissions. The semantics of the Viper language are not formally defined yet. However, they have been partially formalized and characterized [6]. Moreover, most of Viper features have a clear role, and they are explained in an online tutorial [1].

We describe in this section some parts of Viper semantics needed for this thesis. These explanations are partly based on [6], and partly based on the online tutorial.

2.2.1 Permissions and assertions

Permissions. Viper, based on implicit dynamic frames, separates the permission and the value of a heap location. Holding *at least* some fractional permission p to a heap location `x.f`, where x is a reference and f a field, is encoded by the assertion **acc**`(x.f, p)`, whereas the value of $x.f$ being v is encoded by the assertion `x.f == v`. Viper assertions do not use the usual conjunction $\wedge$, they only use the separating conjunction, written `&&` in Viper. As an example, the assertions **acc**`(x.f, 1/2) && x.f == 5` is satisfied by a state which holds *at least* half permission to $x.f$, and whose partial heap defines *at least* $x.f$, and maps the heap location $x.f$ to the value 5.

Assertions. An assertion of the shape **acc**`(x.f, p)` is called an *accessibility predicate*. Assertions which do not contain any accessibility predicates are called pure assertions. Viper assertions can be constructed by combining assertions with the separating conjunction `&&`, the disjunction `||`, and the implication `==>` among others. In particular, disjunctions are only allowed when both assertions are pure. As an example, **acc**`(x.f, 1/2) || x.f == 5` is not a valid assertion in Viper. The left side of an implication also has to be a pure assertion.

Self-framing assertions. Assertions used as annotations, that is as loop invariants and method preconditions and postconditions, have to be self-framing. Simply put, an assertion is self-framing if and only if includes permissions to at least the locations it reads. A better and more formal definition is given in [20]. As an example, **acc**`(x.f, 1/2) && x.f == 5` is self-framing, whereas `x.f == 5` is not.

2.2.2 Viper state model

A Viper state is defined in [6] as a triple of a store, a global heap, and a permission mask. A permission mask is simply a mapping from heap

locations to permission amounts. This permission mask actually consists of three permission masks, for permissions to field locations, permissions to predicates [8], and permissions to magic wands [23].

For our purpose, we only present here a simplified version of a Viper state. We do not consider predicates and magic wands. Moreover, we do not consider global heaps but only local heaps.

For our purpose, a Viper state can be seen as a triple (s, π, h) where:

1. s is the store: A finite partial mapping from local variables to values.
2. π is a permission mask: A mapping from field locations (a pair of a reference and a field) to permissions (elements of $\mathbb{Q} \cap [0, 1]$).
3. h is a local heap: A finite mapping from field locations to values. In particular, the domain of definition of this heap is the finite set of field locations with a permission strictly greater than zero.

Trace semantics and state semantics. Viper semantics is actually a trace semantics. In particular, Viper defines the statement `label l` (where l is simply a name) which records the heap at this point of the execution. The expression `old[l](e)` (where l is a label and e an expression) evaluates the expression e in the heap recorded at the point of the execution the label l refers to. In most of this thesis, we simplify the matter by only considering a state semantics, that is disallowing the use of `label` and `old` features. Chapter 8 discusses how to go from a state semantics to a trace semantics.

2.2.3 Building blocks of loops and method calls verification

We introduce three kinds of statements in this subsection: *havoc*, *inhale* and *exhale*. They are necessary to describe the semantics of loops and method calls in Viper.

Havoc. To havoc a variable simply means to assign an arbitrary value to this variable. As an example, take a state which satisfies $x == 5$, that is its store maps x to 5. If we havoc x in this state, then we "forget" the value of x. x has therefore a new value, on which we do not have any information.[2]

Inhale. The Viper language has an `inhale` statement. To inhale an assertion means to assume the pure assertions (the value constraints) in it, and to add the permissions denoted in the assertion (with accessibility predicates) to the state. As an example, `inhale acc(x.f, 1/2) && x.f == 5` adds half permission to $x.f$ in the permission mask of the state, and assumes that the

[2] Even if loops and method calls are encoded using *havoc* in Viper, *havoc* is not a Viper statement.

value of $x.f$ in the local heap is 5 (that is, if $x.f \neq 5$ in the program state, then the execution stops, and the rest of the program verifies).

Exhale. **`exhale`** is in a way the opposite of **`inhale`**. To exhale an assertion means to assert (verify that it is true, otherwise throw an error) the pure assertions in it, and to subtract the permissions denoted in the assertions from the permission mask of the program state. As an example, **`exhale acc`**`(x.f, 1/2) && x.f == 5` verifies if $x.f == 5$ holds and if the permission mask has at least half permission to $x.f$. If this **`exhale`** verifies, then half permission to $x.f$ is removed from the state.

2.2.4 Black-box semantics of loops in Viper

The general shape of a while loop in Viper is as follows, where b (the guard), I (the loop invariant) are assertions, and s (the loop body) is a statement.

Listing 2.1: General shape of a while loop in Viper

```
1 while (b)
2     invariant I
3 {
4     s
5 }
```

To be well-defined, the loop invariant I should be self-framing, meaning that I must include permissions to at least the locations it reads. Moreover, I is not allowed to contain any permission introspection (see Section 2.2.6). In particular, I cannot use **`perm`** and **`forperm`**.

The verification of this while loop is then equivalent to verifying what we call here the *external behavior of the loop* and the *internal behavior of the loop*. The *internal behavior of the loop* refers to the verification that the loop invariant is actually an invariant for the loop. The *external behavior of the loop* refers to the program, where the loop is approximated using the loop invariant.

The verification of the *external behavior of the loop* proceeds as follows (with respect to the scope surrounding the loop):

1. Exhale the loop invariant: **`exhale`** `I`.
2. Havoc (ie. assign arbitrary values to) all variables that get assigned by the loop body s.
3. Inhale the loop invariant: **`inhale`** `I`.
4. Assume the guard is false: **`assume`** `!b`.

The verification of the rest of the program then proceeds from this state. However, we need to make sure that the loop respects its contract, namely

that executing the loop body s from a state satisfying b and I then satisfies I, and this for any loop iteration, this is why we *havoc* the variables assigned by the loop body. The verification of the *internal behavior of the loop* proceeds as follows:

1. Begin with a state with no permission, only variables defined before the loop and their values are known.
2. Havoc (assign arbitrary values to) all variables that get assigned by the loop body s.
3. Inhale the loop invariant: `inhale I`.
4. Assume the guard: `assume b`.
5. Execute the loop body: `s`.
6. Make sure that the invariant is satisfied, namely exhale it: `exhale I`.

The while loop verifies if and only if both the *internal behavior of the loop* and the *external behavior of the loop* verify.

2.2.5 Semantics of method calls in Viper

Methods and method calls in Viper are verified in a modular way. Methods are verified on their own, and method calls are verified assuming that all methods verify.

Listing 2.2: General shape of a method in Viper

```
method m(arg: T1, ...)
    returns (ret: T2, ...)
    requires P
    ensures Q
{
    s
}
```

A method can take several arguments, specified with their types ($T1$ is for example the type of the argument *arg*), and can return any fixed number of variables, with their types (here *ret* of type $T2$). P is the precondition of the method, whereas Q is its postcondition. Finally, s is the method body. To be well-defined, a method declaration must at least respect the following rules:

1. The method body s is not allowed to modify the value of the arguments (only to read them).
2. The precondition and postcondition, similarly to loop invariants, have to be self-framing, and are not allowed to use permission introspection (see Section 2.2.6 below).

3. The only variables which can be referred to in the precondition P are the arguments.
4. The only variables which can be referred to in the postcondition Q are the arguments and the return variables.
5. The arguments and the return variables have to be mutually distinct.

A method is verified as follows, beginning with an empty state:

1. The arguments and the return variables are added to the store of the empty state, with *some* values.
2. Assume that the state satisfies the precondition: `inhale P`.
3. Execute the method body: `s`.
4. Ensure that the final state satisfies the postcondition: `exhale Q`.

A method call is verified modularly, assuming that the method it refers to verifies. A method call in Viper is written as follows:

Listing 2.3: General shape of a method call in Viper

```
1 y, ... := m(x, ...)
```

It is well-defined if the arguments and return variables match the ones declared (number and types). Informally (we use dots to represent other variables), this method call is equivalent to:

1. (Informal) Make sure the renamed method precondition is satisfied: `exhale P[x.../arg...]`[3]
2. Havoc the return variables $y, ...$
3. (Informal) Assume the renamed method postcondition is satisfied: `inhale Q[x.../arg...][y.../ret...]`

2.2.6 Features which break the frame rule

The permission evaluation feature. As explained in the introduction, Viper has a feature of permission evaluation, `perm`. $perm(x.f)$ evaluates to the permission amount held to $x.f$ by the program state. As an example, the statement `assert perm(x.f) <= 1/2` verifies if and only if the program state holds at most half permission to $x.f$. This statement breaks the frame rule in the following sense:

1. The program
   ```
   inhale acc(x.f, 1/2)
   assert perm(x.f) <= 1/2
   ```

[3] $P[x/arg]$ substitutes arg by x in P.

`exhale acc(x.f, 1/2)`
verifies with an initial empty state (which holds no permission to any field).

2. However, the program
`inhale acc(x.f, 1/2) && acc(x.f, 1/2)`
`assert perm(x.f) <= 1/2`
`exhale acc(x.f, 1/2) && acc(x.f, 1/2)`
does not verify with an initial empty state.

The first program corresponds to the Hoare triple

$$\{\texttt{acc(x.f, 1/2)}\}$$
$$\texttt{assert perm(x.f) <= 1/2}$$
$$\{\texttt{acc(x.f, 1/2)}\}$$

whereas the second program corresponds to the Hoare triple

$$\{\texttt{acc(x.f, 1/2) \&\& acc(x.f, 1/2)}\}$$
$$\texttt{assert perm(x.f) <= 1/2}$$
$$\{\texttt{acc(x.f, 1/2) \&\& acc(x.f, 1/2)}\}$$

The frame rule is broken with this statement, otherwise the second program should verify with $R := acc(x.f, 1/2)$.

For-perm quantifiers. A forperm statement, such as

```
forperm x: Ref [x.f] :: false
```

asserts that, for all references x such that *some* (> 0) permission is held to $x.f$, `false` should hold. This statement verifies if and only if there exists no reference x such that the program state holds some permission to $x.f$. In this case, adding some permission to some field location $x.f$ makes the statement not verify anymore. The frame rule is thus broken with this statement.

Assuming an impure assertion. To assume an assertion means doing nothing if the program state satisfies this assertion. However, if this assertion is not satisfied, the execution stops and the rest of the program verifies. In Viper, it is possible to assume impure assertions, such as `assume acc(x.f, 1/2)`. This statement breaks the frame rule. Indeed, the following Viper statement

```
inhale true
assume acc(x.f, 1/2)
exhale false
```

verifies with an initial empty state, since the assertion is not satisfied (the state has no permission to $x.f$).

However,

```
inhale true && acc(x.f, 1/2)
assume acc(x.f, 1/2)
exhale false && acc(x.f, 1/2)
```

does not verify, since the assumed assertion is satisfied, but **false** does not.

Chapter 3

Unsoundness of Inlining in Viper

As explained in Chapter 1, static inlining (inlining thereafter) is not always sound for Viper programs. In this section, we explore concrete examples where inlining is unsound to derive a *soundness condition*. The *soundness condition* characterizes those Viper programs for which inlining is sound. We first express the *soundness* property and the requirement for the *soundness condition* formally. We then show three examples of unsoundness in Viper, which give rise to three properties of statements (**safeMono**, **monoOut** and **framing**), needed to characterize the *soundness condition*. Finally, we explore these three properties in Viper, and express a *soundness condition* for a simplified case.

3.1 Soundness property and soundness condition

The main purpose of inlining is to detect *fundamental errors*, as explained in Chapter 1. The idea is that if the inlined statement (for any bound) does not verify, then there must exist a fundamental error in the original program.

Definition 3.1 ***Soundness of inlining***
If there exists an annotation A such that the initial statement s, which can call methods from the annotated program $annotate(Pr, A)$, verifies with an initial empty state, then the statement inlined with respect to the program Pr up to the depth bound n $inline_n(s, Pr)$ verifies with an initial empty state.
We denote this property $soundness^n_{Pr}(s)$.

We show in this chapter that this property does not hold in general for Viper programs. We therefore want to find a *soundness condition*, ideally the weakest possible, such that *soundness* holds. We denote this *soundness condition* $SC^n_{Pr}(s)$ for a program Pr, an initial statement s, and a bound n. This *soundness condition* has to satisfy the following requirement:

Listing 3.1: Annotated original program.

```
1  field f: Int
2
3  method initial(x: Ref)
4  {
5      inhale acc(x.f)
6      callee(x)
7      assert perm(x.f) <= 1/2
8  }
9
10 method callee(x: Ref)
11     requires acc(x.f)
12     ensures true
13 {
14     assert true
15 }
```

Listing 3.2: Inlined program (bound of 1).

```
1  field f: Int
2
3  method initial_inlined(x: Ref)
4  {
5      inhale acc(x.f)
6      assert true
7      assert perm(x.f) <= 1/2
8  }
```

Figure 3.1: Example of unsoundness of inlining in Viper: Statement not **safeMono**.

Definition 3.2 *Requirement for the soundness condition*

$$SC^{n}_{Pr}(s) \implies soundness^{n}_{Pr}(s)$$

This expresses that, if the soundness condition holds for a statement, a program, and a bound, then inlining this statement with respect to this program up to this bound is sound. Namely, if it is possible to find an annotation such that the annotated program and the statement s (with respect to the annotated program) both verify, then inlining this statement up to a depth of n verifies. The contraposition of the soundness property is particularly useful: Under the soundness condition, if the inlined statement does not verify (for the bound n), then it is not possible to find an annotation to make the program verify. This means we detected a *fundamental error*, without needing any annotation from the user.

The next parts of this section focus on examples in Viper where inlining is unsound, in order to determine a suitable soundness condition.

3.2 Examples of Viper programs where inlining is unsound

3.2.1 SafeMono: Stronger states verify more

The first example of unsoundness of inlining in Viper, shown in Figure 3.1, is based on permission leaks and permission introspection. The annotated program (Listing 3.1) verifies whereas the inlined program (Listing 3.2) does

not verify. The annotated program verifies because of the permission leak happening when calling the *callee* method. Before the call, full permission is held by the program state to *x.f*, then half of this permission is leaked to the method, which does not return anything. After the call, only half permission is held to *x.f*, which makes line 7 verify.

The inlined program (Listing 3.2) does not verify because of line 7. Line 5 gives full permission to *x.f*, line 6 is the inlined method body (which does nothing), thus full permission is still held on line 7.

Because of possible permission leaks, the state after a method call in an inlined program is often *stronger* than the corresponding state in the annotated program. Let us define this notion of *stronger* and *weaker* states.

Definition 3.3 ***Stronger and weaker Viper states***
A Viper state (s', π', h') *is* ***stronger*** *than another Viper state* (s, π, h) *if and only if*

1. *For all* x *defined in* s*, then* x *is also defined in* s'*, and* $s'(x) = s(x)$*.*
2. *For all references* r *and fields* f*,* $\pi'(r, f) \geq \pi(r, f)$*.*
3. *For all references* r *and fields* f *such that* $\pi(r, f) > 0$*,* $h'(r, f) = h(r, f)$*.*

If the state φ' *is* ***stronger*** *than* φ*, then* φ *is* ***weaker*** *than* φ'*.*

A set of states A' *is* ***stronger*** *than a set of states* A *if and only if, for all states* $\varphi' \in A'$*, there exists a state* $\varphi \in A$ *such that* φ' *is stronger than* φ*.*[1]

This *stronger* relation for states means two things:

1. The stronger state has at least the information contained in the weaker state, namely the store values and the heap values.
2. The stronger state holds at least as much permission to heap locations as the weaker state.

We can now clarify why inlining the program in Figure 3.1 is unsound. The issue comes from the statement `assert perm(x.f) <= 1/2`. Indeed, there exist situations in which a weaker state makes this statement verify, whereas a stronger state makes it fail. Such a statement does not have the **safeMono** property:

Definition 3.4 *A statement* s *is* ***safeMono*** *if and only if, for any two states* φ *and* φ' *such that* φ' *is stronger than* φ *and* s *verifies with the state* φ*, then* s *also verifies with the state* φ'*.*

This property has already been described in [25] and [5], where it is called *Safety Monotonicity*. However, this property is not enough to express the soundness condition, as shown by the next example.

[1] This order relation on sets of states is not antisymmetric. As an example, take φ a state with full permission to $x.f$, and φ' a state with half permission to $x.f$. Then $\{\varphi, \varphi'\}$ is both stronger and weaker than $\{\varphi'\}$.

Listing 3.3: Annotated original program.

```
1  field f: Int
2
3  method initial(x: Ref)
4  {
5      inhale acc(x.f)
6      callee(x)
7      if (perm(x.f) >= 1/1) {
8          exhale acc(x.f)
9      }
10     callee(x)
11 }
12
13 method callee(x: Ref)
14     requires acc(x.f, 1/2)
15     ensures true
16 {
17     assert acc(x.f, 1/2)
18 }
```

Listing 3.4: Inlined program (bound of 1).

```
1  field f: Int
2
3  method initial(x: Ref)
4  {
5      inhale acc(x.f)
6      assert acc(x.f, 1/2)
7      if (perm(x.f) >= 1/1) {
8          exhale acc(x.f)
9      }
10     assert acc(x.f, 1/2)
11 }
```

Figure 3.2: Example of unsoundness of inlining in Viper: Statement not **monoOut**.

3.2.2 MonoOut: Stonger state in, stronger states out

The example shown in Figure 3.2, like the previous one, is based on permission leaks and permission introspection, but this time with two method calls instead of one. As before, the annotated program (Listing 3.3) verifies whereas the inlined one (Listing 3.4) does not. The first call to *callee* in the annotated program is made with full permission to *x.f*. The method call verifies since there is at least half permission to *x.f*, and leaks half permission. After this first method call, only half permission is held, therefore the state doesn't enter the *if* branch, and goes directly to the second method call. Once again, this method call verifies since there is at least half permission held.

On the other hand, full permission to *x.f* is still held in the inlined program after the inlined method call, on line 6. The *if* branch is therefore entered, where full permission is exhaled. The statement on line 10 therefore doesn't verify, since no permission is held anymore.

Notice that the statement `if (perm(x.f) >= 1/1) {exhale acc(x.f)}` (lines 7 to 9) is **safeMono**: Indeed, this statement always verifies. Hence, as this example shows, the **safeMono** property is not enough to guarantee soundness of inlining. The issue is that a stronger state before the statement (full permission to $x.f$ for example, compared to half permission) can result in a weaker state after the statement (no permission compared to half permission). Such a statement does not have the **monoOut** property.

Listing 3.5: Annotated original program.

```
1  field f: Int
2
3  method initial(x: Ref)
4  {
5      inhale acc(x.f)
6      callee(x)
7      x.f := 5
8  }
9
10 method callee(x: Ref)
11     requires true
12     ensures true
13 {
14     exhale acc(x.f, perm(x.f))
15 }
```

Listing 3.6: Inlined program (bound of 1).

```
1  field f: Int
2
3  method initial(x: Ref)
4  {
5      inhale acc(x.f)
6      exhale acc(x.f, perm(x.f))
7      x.f := 5
8  }
```

Figure 3.3: Example of unsoundness of inlining in Viper: Statement not **framing**.

Definition 3.5 *A statement s is **monoOut** if and only if, for any two states φ and φ' such that φ' is stronger than φ and s verifies with both states, then executing s from the state φ' results in a stronger set of states than executing it from the state φ.*

Combining both properties results in **mono**.

Definition 3.6 *A statement s is **mono** if and only if it is **safeMono** and **monoOut**.*

However, even if all statements and substatements of all methods of the program and of the initial statement are **mono**, we still do not have soundness.

3.2.3 Framing: Statements which respect the frame rule

The two previous examples showed us some conditions needed on substatements of the initial statement, but do not say anything about the body of a method which is called. In the example shown in Figure 3.3, the body of *callee* is **mono**. Indeed, it always verifies, and results in a state identical as the one before the statement, except that there is no permission to $x.f$.

The annotated program (Listing 3.5) verifies because it does not transfer any permission to *callee*, thus it still has full permission on line 7, and full permission is needed to write the value of $x.f$. On the other hand, the body of *callee* on line 6 of the inlined program (Listing 3.6) removes all permission to $x.f$, hence making line 7 fail. The underlying reason is that the idea of modular verification of methods derives from the frame rule (see Section 2.1), which not all statements satisfy in Viper, due to permission introspection.

Definition 3.7 *(Informal) A statement is **framing** if and only if it is **mono** and it respects the frame rule.*

Intuitively, a statement s is **framing** if it satisfies the following: "If we add information and permissions to a state which already verifies with the statement s, then we still have these additional information and permissions after the statement has finished executing". This view is of course a bit informal, since we need to make sure there is no interference between the information provided and the statement. For example, if we add information about the value of a variable x and s modifies x, then we lose this information after the statement. Another way to say it is that a **framing** statement only acts on a small part of a state, and does not "consume" the rest of it. However, we allow a **framing** statement to add more than what we added initially. As an example, a statement which doubles the permission held to $x.f$ (such as `inhale acc(x.f, perm(x.f))` is **framing**. This property has also been called *Framing Property* [25, 5]. A formal definition is given in Chapter 6.

These three examples give us a first impression of a general **soundness condition**, a sufficient condition for soundness of inlining to hold. Let us first define what it means to be inlinable.

Definition 3.8 *A statement is **inlinable** if and only if it is a loop, a method call, or if it contains an inlinable substatement.*

We have now a first idea of what the soundness condition should look like:

1. Non-inlinable substatements have to be **mono**.
2. Bodies of methods and loops have to be **framing**.

This first idea is really informal and only aims at giving an intuition, since for example we do not refer here to the bound n. The complete soundness condition is formally expressed in Chapter 6. The purpose of the next and last part of this section is to illustrate and explore these properties a bit more, before diving into the formalization.

3.3 Framing and mono

3.3.1 Inclusions and examples

By definition, **mono** is the intersection of **safeMono** and **monoOut**, and **framing** is included in **mono**, as represented in Figure 3.4. Examples to show that these inclusions are strict are given in Table 3.1.

Statement 1 is the sequential composition of statement 2 and statement 3. Statement 2 is not **safeMono**, since it can fail when adding more permission. It is otherwise **monoOut** since it does not modify the state if it verifies. Statement 3 is not **monoOut**, but it is **safeMono** since it always verifies.

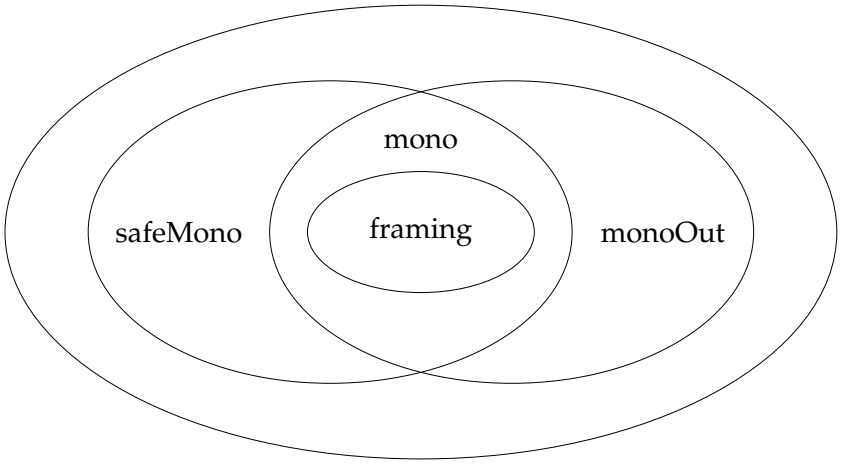


Figure 3.4: Representation of sets of statements satisfying **safeMono**, **monoOut** and **framing**.

Table 3.1: Examples of **safeMono**, **monoOut** and **framing** statements.

#	Statement	**safeMono**	**monoOut**	**framing**
1	`assert perm(x.f) <= 1/2` `exhale perm(x.f) >= 1/1 ==> acc(x.f)`			
2	`assert perm(x.f) <= 1/2`		✓	
3	`exhale perm(x.f) >= 1/1 ==> acc(x.f)`	✓		
4	`exhale acc(x.f, perm(x.f))`	✓	✓	
5	`exhale acc(x.f)`	✓	✓	✓

Because of these properties, statement 1 is neither **safeMono** nor **monoOut**. Statement 4 is **mono** but not **framing**, as explained before. Finally, statement 5 is **framing** since it respects the *frame rule*.

3.3.2 Compositionality

safeMono is not stable by sequential composition. As an example, take $s_1 :=$ `exhale perm(x.f) >= 1/1 ==> acc(x.f)`, and $s_2 :=$ `assert acc(x.f, 1/2)`. Both s_1 and s_2 are **safeMono**, but $s_1 \,;\, s_2$ is not: A state with half permission to $x.f$ verifies, but a stronger state with full permission to $x.f$ fails.

mono (and **monoOut**) are stable by sequential composition. Indeed, take $s := s_1 \,;\, s_2$ with s_1 and s_2 **mono**. Take two states φ and φ' such that φ' is stronger than φ and φ verifies with s (thus φ verifies with s_1). Since s_1 is **mono**, we have that it verifies with φ' too, and that executing it with the initial state φ' results in stronger states than executing it with the initial state φ. By using that s_2 is **mono**, we can show that s is **mono**.

This reasoning can be extended for **framing**, and for non-deterministic if (`if (*)`):

Lemma 3.9 *Assume that s_1 and s_2 are **mono** (resp. **framing**). Then*

1. *s_1 ; s_2 is **mono** (resp. **framing**).*
2. *`if (*) {s1} else {s2}` is **mono** (resp. **framing**).*

This result is particularly useful in practice. As shown in Chapter 6, it is necessary to determine whether some statements are **framing** and **mono** to guarantee soundness. This result means that showing that substatements of a statement are **mono** or **framing** is sufficient to prove that this statement is **mono** or **framing**.

However, the converse also does not hold. As an example, consider

$s_1 :=$ `var p: Perm := perm(x.f); exhale acc(x.f, p - 1/2)`
$s_2 :=$ `inhale perm(x.f) <= 1/2 ==> acc(x.f, p - 1/2)`
$s :=$ s_1 ; s_2

s_1 and s_2 are not **framing**, but s_1 ; s_2 is. Indeed, s_1 records the old permission held to $x.f$ in p, and then removes enough permission to $x.f$ to reach half permission (it fails if there is less than half permission). Therefore, any state after s_1 has half permission to $x.f$. s_2 makes the permission to $x.f$ increase by $p - \frac{1}{2}$ if the permission held to $x.f$ is lower than half, therefore s_2 is not **framing** (actually not even **mono**, take for example two states with $p = \frac{1}{2}$, one with $\frac{1}{2}$ permission to $x.f$ and one with $\frac{3}{4}$). Finally, s_1 ; s_2 is **framing** since when it verifies, the state after this statement is the same as the one before.

The converse does not hold also with non-deterministic if. As an example, consider

$s_1 :=$ `exhale perm(x.f) == 1/2 ==> acc(x.f, perm(x.f))`
$s_2 :=$ `exhale perm(x.f) != 1/2 ==> acc(x.f, perm(x.f))`
$s :=$ `if (*) {s1} else {s2}`

Both statements always verify, thus they are **safeMono**. However, none of them is **monoOut**, thus they are neither **mono** nor **framing**. The statement s always verifies and creates two states from an initial one: The same one, and a similar one where no more permission to $x.f$ is held. This statement is therefore **monoOut**, thus **mono**.

These compositionality properties mean that we have a sufficient condition to determine whether a statement is **mono** and **framing** by looking only at its substatements, but this is not enough to show that a statement is not **mono** or **framing**. To show that, one has to find explicit counter-examples.

To summarize, we explored examples of Viper programs where static inlining is unsound, in order to sketch a soundness condition. Our aim is to

find and express the weakest condition under which soundness of inlining holds. Through three examples, we exhibited three properties of statements, **safeMono**, **monoOut** (combined in **mono**), and **framing**, which are required to have soundness. These properties can be lifted to more general settings. The next sections define (1) a general *separation algebra* to abstract Viper's state model, and (2) a language abstracting resource-based languages such as Viper. Using these two ingredients, we then formalize and prove soundness under the soundness condition for a more general setting

Chapter 4

Abstracting Viper States with a Separation Algebra

The previous chapter focused on exploring examples of Viper programs where inlining is unsound, in order to find a general soundness condition. This chapter and Chapter 5 define a general model for a verification language, based on Viper. The idea is to lift the results obtained in Chapter 3. This chapter abstracts the Viper state model, using the concept of separation algebra [5]. Chapter 5 then defines a general parametric language and a semantics for this language, using the algebra from this chapter to represent states. Finally, Chapter 6 formally defines the **mono** and **framing** properties in this framework. Using these properties, we formally express a soundness condition, and prove that it implies soundness of inlining. The content and the proofs presented in this chapter, Chapter 5 and Chapter 6 have been mechanized and proved using the proof assistant Isabelle/HOL [16] (see the formalization in Appendix A).

A state in Viper contains resources, and more precisely two kinds of resources, *pure* and *impure* resources. The store contains *pure* resources, in the sense of duplicable resources. As an example, the assertion `n == 5` is equivalent to the assertion `n == 5 && n == 5`. On the other hand, permissions to heap locations are no duplicable: Permissions are impure resources. As an example, the assertion **acc**`(x.f, 1/2)` is not equivalent to the assertion **acc**`(x.f, 1/2) &&` **acc**`(x.f, 1/2)`. We define in this chapter a separation algebra, whose aim is to abstract the Viper state model. A separation algebra [5] defines what it means to *add* two states, and defines an order relation based on this addition. The algebra we define here makes a clear distinction between *pure* resources, namely the store, and *impure* resources, corresponding to the permission mask and the local heap in Viper. In that respect, it does not correspond exactly to a separation algebra (as in [5, 7]) but it can be seen as a combination of a separation algebra with a pure part (the store).

The plan of this chapter is as follows: We first define a preliminary separation algebra, the basis of our separation algebra. We then describe, using this preliminary separation algebra, the behavior of the Viper statements **`inhale`** and **`exhale`**, restricted to self-framing assertions. These statements constitute the core of our framework, since they are the blocks at the basis of the semantics of loops and method calls. We then fully define the separation algebra, and finally highlight differences with related works.

4.1 A preliminary separation algebra

We first define a preliminary separation algebra, to be able to speak about addition of states as well as stronger and weaker states in an abstract way.

Definition 4.1 *A **preliminary separation algebra** is a partial commutative monoid $(\Sigma, \oplus, u)$, where Σ is a set of states, $\oplus$ (addition) is a partial binary operation $\Sigma \times \Sigma \to \Sigma$, and u (neutral state) is an element of Σ with the following properties, for all states $a, b, c \in \Sigma$ (we note $a\#b$ if and only if $a \oplus b$ is defined):*

1. *Commutativity (definedness): $a\#b \Leftrightarrow b\#a$*
2. *Commutativity: $a\#b \Rightarrow a \oplus b = b \oplus a$*
3. *Associativity (definedness): $a\#b\#c \Leftrightarrow a\#b \wedge (a \oplus b)\#c \Leftrightarrow b\#c \wedge a\#(b \oplus c)$*
4. *Associativity: $a\#b\#c \Rightarrow a \oplus (b \oplus c) = (a \oplus b) \oplus c$*
5. *Neutral state: $a\#u$ and $a \oplus u = a$*

This definition is similar to the one from [5], with one major difference: We do not require $\oplus$ to be cancellative (that is, $\lambda y.x \oplus y$ is not necessarily injective). The reason is that we also want to include pure resources in the states, and pure states are duplicable (by definition), which therefore makes this requirement unsatisfiable.

Definition 4.2 *A state φ is **pure** if and only if $\varphi \oplus \varphi = \varphi$.*

One way to understand this definition is that a state is pure if and only if it is infinitely duplicable. This definition allows us to extract from a state its *core*, namely its maximum pure part. To define the core of a state, we first need to define a partial order on states:

Definition 4.3 ***Induced order on states**: $\varphi_1 << \varphi_2 \iff \exists \varphi'.\varphi_2 = \varphi_1 \oplus \varphi'$*

Lemma 4.4 ***The induced order is a partial order**:*

1. *$<<$ is reflexive.*
2. *$<<$ is transitive.*
3. *$\forall \varphi \in \Sigma.u << \varphi$.*

Listing 4.1: Pure assertion.

```
field f: Int

method m(n: Int, x: Ref)
{
    inhale n >= 0
    exhale n >= 0
    assert n >= 0
}
```

Listing 4.2: Accessibility predicate.

```
field f: Int

method m(n: Int, x: Ref)
{
    inhale acc(x.f)
    exhale acc(x.f)
    assert perm(x.f) == 0/1
}
```

Listing 4.3: Combination of a pure assertion and an accessibility predicate.

```
field f: Int

method m(n: Int, x: Ref)
{
    inhale n >= 0 && acc(x.f)
    exhale n >= 0 && acc(x.f)
    assert n >= 0 && perm(x.f) == 0/1
}
```

Figure 4.1: Inhaling and exhaling the same assertion.

4.2 Motivation: Modeling the behavior of inhale and exhale on annotations

The preliminary separation algebra is the basis of our separation algebra. We explain in this section the reason why we distinguish two kinds of resources, pure and impure. Inlining deals with method calls and loops. Moreover, in Viper, the verification of loops and method calls is based on two statements, **inhale** and **exhale**, for self-framing (as defined in [20]) assertions. In this section, we illustrate the behavior of **inhale** and **exhale** with pure assertions and assertions containing accessibility predicates. Annotations in Viper, such as loop invariants, can be seen as having two roles: One role, based on pure resources, is simply to make the approximation of the behavior of the loop more precise. The second role, based on impure resources, is to model the transfer of ownership of heap locations.

In Figure 4.1 are presented three programs, all of whom verify. These three programs begin with an almost empty state (only two variables, n and x, are defined). They then **inhale** and **exhale** the same assertion. The purpose of the assertions in the end is to show what we know about the final states.

The first example (Listing 4.1) inhales and exhales a pure assertion, that is an assertion speaking about a pure resource (the local variable n). **inhale** in the

case of a pure assertion is equivalent to **assume**, namely only the states which satisfy this assertion go through, the execution is stopped for the other ones. **exhale** in the case of a pure assertion is equivalent to **assert**, which verifies if and only if the assertion is true. Therefore, inhaling and exhaling a pure assertion is equivalent to assuming this assertion, and the **exhale** statement does not modify the state.

In the second example (Listing 4.2), the assertion is an accessibility predicate. Inhaling this assertion means adding the permission it contains to the current state, namely going from no permission to $x.f$ to full permission to $x.f$. Exhaling this assertion then "cancels" this by removing the permission contained in the assertion, namely going from full permission to no permission. This is a fundamental difference from the previous example, where pure information is not forgotten.

The last example (Listing 4.3) shows what happens when pure and accessibility predicate are combined. The pure information ($n \geq 0$) stays, whereas the permission is removed. This difference is at the core of our separation algebra.

4.3 Definition of the separation algebra

To finally define the separation algebra, we need to define the core of a state, which represents the pure part of the state, namely the store.

Definition 4.5 ***Core of a state***
The core set of a state is defined as

$$coreset(\varphi) := \{p \in \Sigma . p \text{ is pure} \wedge p << \varphi\}$$

The core of a state φ is denoted $|\varphi|$, and is defined as[1]

$$|\varphi| := \sum_{p \in coreset(\varphi)} p$$

Lemma 4.6 ***Core properties****:*

1. $u \in coreset(\varphi)$
2. $coreset(\varphi)$ *is stable by* $\oplus$*:* $\forall a, b \in coreset(\varphi).a\#b \wedge a \oplus b \in coreset(\varphi)$
3. *If* $coreset(\varphi)$ *is finite, then* $|\varphi| = \max(coreset(\varphi))$

The situation we are interested in is when $coreset(\varphi)$ is finite for all states, namely when all states have a finite store. We now define the separation algebra, using a new function $C : \Sigma \to \Sigma$, which computes the *complementary*

[1] The Σ symbol refers to a sum using $\oplus$.

state of the core of a state. The core of a state $|\varphi|$ contains the pure resources of the state, whereas the complementary $C(\varphi)$ contains the impure resources of the state (permissions to and values of heap locations):

Definition 4.7 *A **separation algebra** is a quadruple $(\Sigma, \oplus, u, C)$ which satisfies the following properties:*

1. *$(\Sigma, \oplus, u)$ is a preliminary separation algebra.*
2. *Finiteness: $\forall \varphi \in \Sigma.coreset(\varphi)$ is finite.*
3. *Partially cancellative: $\forall \varphi, a, b \in \Sigma.C(\varphi) \oplus a = C(\varphi) \oplus b \implies a = b$*
4. *Decomposition: $\forall \varphi \in \Sigma.\varphi = |\varphi| \oplus C(\varphi)$*
5. *Empty core: $\forall \varphi \in \Sigma.|C(\varphi)| = u$*
6. *Uniqueness: $\forall \varphi, a \in \Sigma.\varphi = |\varphi| \oplus a \wedge |a| = u \implies a = C(\varphi)$*
7. *Positivity: $\forall a, b \in \Sigma.a \oplus b = u \implies a = u$*
8. *Pure reducibility: $\forall p, a, b \in \Sigma.p << a \oplus b \wedge p$ is pure $\implies p << |a| \oplus |b|$*

We already presented the first axiom. The purpose of the finiteness axiom is to have the core of a state being the maximum of the *coreset*, which means that all stores are finite. Axiom 3 gives us the cancellative property that we gave up from [5] by allowing pure states to exist. Axioms 4, 5 and 6 ensure that $C(\varphi)$ is the unique state with no pure part such that, added to the core of φ, gives φ. Axiom 7 means that nothing can come out of an empty state. Axiom 8 means that if a pure state is smaller than $a \oplus b$, then its resources are contained in the combination of the pure resources from a ($|a|$) and the pure resources of b ($|b|$).

Remark 4.8 *From the finiteness axiom and Lemma 4.6, we already get that*

$$\forall \varphi \in \Sigma.\exists c \in \Sigma.\varphi = |\varphi| \oplus c$$

without the uniqueness and the empty core axioms.

Remark 4.9 *It is possible to inject the original separation algebra (from [5]) in this separation algebra, by getting rid of the store and of pure states, namely by having that $\forall \varphi \in \Sigma.|\varphi| = u \wedge C(\varphi) = \varphi$. Everything (except positivity) can then be directly proven.*

Here are some useful properties which can be proven from these axioms:

Lemma 4.10 *Suppose $(\Sigma, \oplus, u, C)$ is a separation algebra. Then*

1. *$<<$ is antisymmetric.*
2. *$\forall \varphi_1, \varphi_2 \in \Sigma.\varphi_1 = \varphi_2 \iff |\varphi_1| = |\varphi_2| \wedge C(\varphi_1) = C(\varphi_2)$*

3. $\forall \varphi_1, \varphi_2 \in \Sigma. \varphi_1 << \varphi_2 \iff |\varphi_1| << |\varphi_2| \wedge C(\varphi_1) << C(\varphi_2)$
4. $\forall \varphi \in \Sigma. |\varphi| << \varphi \wedge ||\varphi|| = |\varphi|$
5. $\forall \varphi \in \Sigma. C(\varphi) << \varphi \wedge C(C(\varphi)) = C(\varphi)$

Finally, we define an addition and an order on sets, which formalize the corresponding definitions from Chapter 3.

Definition 4.11 ***Addition and order on sets***
Let $A, B \subseteq \Sigma$. Then

$$A \oplus B := \{a \oplus b | a \in A, b \in B, a\#b\}$$

$$A >> B \iff (\forall a \in A. \exists b \in B. b << a)$$

Lemma 4.12 ***Properties of addition of sets***

1. *Commutativity: $A \oplus B = B \oplus A$.*
2. *Associativity: $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.*
3. *Transitivity: $A >> B \wedge B >> C \implies A >> C$.*
4. *Neutral: $A \oplus \{u\} = A$.*
5. *Transitive closure: $A >> B \iff \{a' | \exists a \in A. a << a'\}. \subseteq \{b' | \exists b \in B. b << b'\}$.*
6. *Biggest element: $\varnothing >> A$.*
7. *Addition is bigger: $A = B \oplus C \implies A >> B$.*

Remark 4.13 *The reciprocal of property 7 is not true. As an example, take three states x_f, x_h, y_f where x_f (resp. y_f) has full permission to $x.f$ (resp. $y.f$) and x_h has half permission to $x.f$.*

Then $A := \{x_h \oplus y_f, x_f\} >> B := \{x_h, x_f\}$. Now imagine there exists C such that $A = B \oplus C$. We have $y_f \in C$, thus $x_f \oplus y_f \in B \oplus C$, but $x_f \oplus y_f \notin A$.

4.4 Related work

The concept of abstract separation logic and separation algebra was introduced in 2007 [5]. A separation algebra is defined in this work as a "cancellative, partial commutative monoid". Separateness and substate relations are induced from this separation algebra. We based our work on these separation algebra axioms and these induced relations. However, we do not consider the same cancellative axiom. As we explained in this section, we want to consider duplicable states (pure states) representing stores in our algebra.

This then implies, by definition, that our algebra cannot be cancellative. We still require a cancellative property, but only for states with empty stores.

A "fresh look at separation algebras" was presented in 2009 [7]. In this work, the authors begin with the same axioms as before. They then weaken the existence of a neutral element (*the* unit) axiom with the existence of *a* unit for all elements. They go from

$$\exists u. \forall x. u \oplus x = x$$

to

$$\forall x. \exists u_x. u_x \oplus x = x$$

These units can be seen in a way as corresponding to pure states and cores (since $|\varphi| \oplus \varphi = \varphi$), even though the existence of a neutral state in our algebra makes this axiom trivially satisfied.

They also add some more axioms. A positivity axiom

$$a \oplus b = c \wedge c \oplus c = c \Longrightarrow a \oplus a = a$$

that we reproduce in a way. This axiom, directly intepreted in our algebra, means that if adding two states results in a state with only a store (and no permission), then both states are pure (only stores and no permission). We have a similar axiom (also called "positivity", Definition 4.7) saying that one can only obtain empty states from the empty state. These axioms ensure that the induced relation is antisymmetric.

They also add a disjointness axiom (stronger than the positivity one), requiring that only pure states can be joined (added) to themselves. We do not use this axiom since Viper uses fractional permissions, because fractional permissions do not make the difference between two equivalent amounts of permission. Indeed, two Viper states with permission to read the same field can be joined as long as the sum of their permissions is at most one (the full permission).

Finally, they consider a splittability axiom requiring that it is always possible to split a non-empty state in two non-empty states, and the following cross-split axiom:

$$\begin{aligned} &a \oplus b = z \wedge c \oplus d = z \\ &\Longrightarrow \exists ac, ad, bc, bd. ac \oplus ad = a \wedge bc \oplus bd = b \wedge ac \oplus bc = c \wedge ad \oplus bd = d \end{aligned}$$

We do not use these axioms, but use a similar one (pure reducibility, Definition 4.7) requiring that if a pure state (a store) is a substate of the sum of two states, then this pure state is smaller than the sum of the cores of the summands.

Finally, we use the same name and notation for the core as Iris [11]. In this work, the authors define a resource algebra with a duplicable core noted $|\_|$, which has properties similar to our own. This core is duplicable ($a \oplus |a| = a$), idempotent ($||a|| = |a|$), and monotone ($a << b \implies |a| << |b|$), similarly to ours. However, the core operator they consider is only partial, whereas ours is always defined.

To conclude this chapter, we defined a general separation algebra to capture the key properties of the Viper state model as well the behaviour of **inhale** and **exhale** statements on self-framing assertions. In particular, we define a separation algebra which considers impure as well as pure (duplicable) resources. We first presented a simple preliminary separation algebra on which we build on. We then showed, through examples, the motivation for the shape of our separation algebra, based on the behavior of **`inhale`** and **`exhale`** statements, statements which are at the basis of loops and method calls. Finally, we defined this separation algebra with several axioms, showed some interesting properties, and compared it to related works.

The next chapter builds a language on top of this separation algebra. The purpose of this language is to be as general as possible while modeling the semantics of Viper loops and method calls, in order to show general soundness and completeness properties for inlining.

Chapter 5

A Parametric Language for Resource-Based Inlining

In this chapter, we define a general parametric language for studying resource-based inlining. The states for the semantics of this parametric language are elements of a separation algebra, as defined in Chapter 4. The structure of the model is represented in Figure 5.1. The inputs for this model consist in a separation algebra, two types, and four interfaces. Using these inputs, we define a language and its semantics. We make this language as general as possible, by just requiring the minimum needed for reaching two objectives:

1. Model as closely as possible the semantics of loops and method calls in Viper.
2. Express a *soundness condition* as sketched in Chapter 3 and prove *soundness* of inlining.

This chapter is organised as follows. We fix a separation algebra $(\Sigma, \oplus, u, C)$ throughout the chapter. Section 5.1 presents the parameters of the model, the statements of the language, and the general shape of its semantics. Section 5.2 then describes the type of variables, the store interface, and defines the semantics of defining and havocing variables. Section 5.3 explores how to encode syntactic assertions from Viper into semantic assertions, and defines the role of the assertions interface, the semantics of inhaling, exhaling and assuming, all needed for encoding loops and method calls. Section 5.4 describes the semantics of while loops. Section 5.5 describes the need for and the role of the rename interface, as well as its requirements. Section 5.6 describes the semantics of method calls, using the previously defined renaming functions. Section 5.7 completes the definition of the semantics, by describing the role of the type of *custom* statements, and the requirements for the custom interface. Finally, Section 5.8 describes what it means for a program to be well-defined and to verify. Everything presented

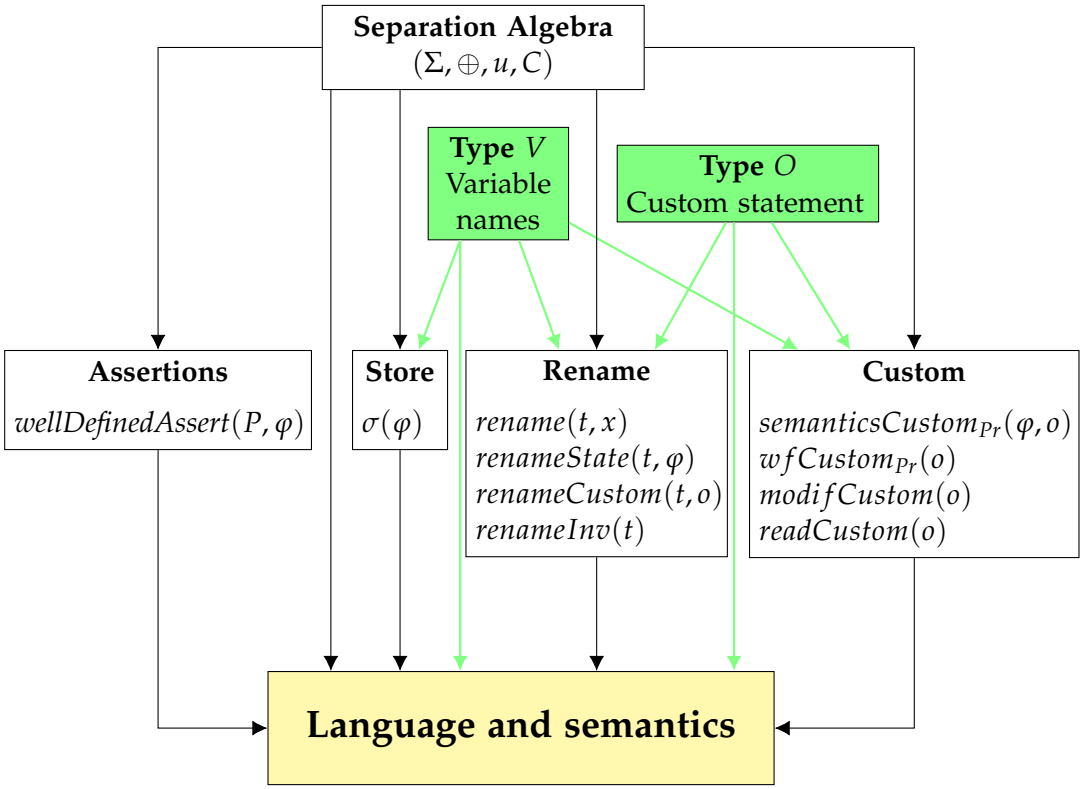


Figure 5.1: Illustrations of the input parameters for the model. P is a semantic assertion, φ is an element of Σ, t is a renaming quadruple, and o is a *custom* statement of type O.

in this chapter has been mechanized and proved with the proof assistant Isabelle/HOL [16] (see Appendix A).

5.1 The language

5.1.1 Parameters of the model

Figure 5.1 represents all the input parameters of our framework. Based on these parameters, our framework defines a language and its semantics. This model requires first a separation algebra, whose elements are the states for the semantics. On top of this separation algebra, two types are required:

1. **Type** V is the type of variable names in our language (*string* in Viper).
2. **Type** O is a type of *custom* statements. This allows the model to be instantiated with statements that are not defined initially in the statements of our language. As an example, our language does not define an **`assert`** statement, and does not define a statement for assigning variables. These two statements can therefore be defined through this

type O (see Chapter 8 for more details).

The remaining parameters are four interfaces. These interfaces have to provide the functions listed on the figure. Moreover, these functions must satisfy some requirements. These requirements are given throughout the chapter.

The first interface, the **store** interface, simply defines a store function σ, such that

$$\sigma(\varphi)$$

is the set of the names of the variables defined in the state φ. Section 5.2 explains this.

The **assertions** interface defines a function, whose aim is to model the well-definedness of an assertion in a given state. The function

$$wellDefinedAssert(P, \varphi)$$

returns a boolean: true if the semantic assertion P is well-defined in the state φ, false otherwise. It is explained in Section 5.3.

A method call $\vec{y} := m(\vec{x})$ requires to rename the formal arguments of the methods to $\vec{x}$, and the result variables of the method to $\vec{y}$. This is one role for the **rename** interface. The other role is to rename method bodies when inlining, as defined in Chapter 6. This interface should provide four functions:

1. $rename(t, x)$ (variable names).
2. $renameState(t, \varphi)$ (states).
3. $renameCustom(t, o)$ (*custom* statements).
4. $renameInv(t)$ (inverts a renaming quadruple).

where t is a renaming quadruple (a configuration with four lists of variable names which defines how to rename). *rename* renames a variable name to another, *renameState* renames a state to another state (that is it renames the variables in the store of the state), *renameCustom* renames a *custom* statement of type O to another. Finally, *renameInv* takes a renaming quadruple and returns another, which inverts the renaming. The meaning of these functions and the requirements they should satisfy are explained in Section 5.5.

Finally, the **custom** statements allowed in our language are characterized by the following four functions:

1. $semanticsCustom_{Pr}(\varphi, o)$ (semantics).
2. $wfCustom_{Pr}(o)$ (well-formed statement).

3. $modifCustom(o)$ (variables modified by a statement).
4. $readCustom(o)$ (variables read by a statement).

These functions are also explained in Section 5.7.

The following requirements deal with the **rename** interface:

1. Requirement 5.37: renaming an element.
2. Requirement 5.38: renaming a state.
3. Requirement 5.40: renaming an assertion.
4. Requirement 5.48: renaming a *custom* statement.

Finally, the following requirements deal with the other interfaces:

1. Requirement 5.6: **store** interface.
2. Requirement 5.23: **assertions** interface.
3. Requirement 5.47: **custom** interface.

5.1.2 Statements

The statements of the language presented in this chapter are as follows:

Definition 5.1 ***Statements of the language***
S represents statements, A assertions, $\vec{V}$ lists of elements of V (variable names) and O is the type of custom statements.

$$S := S\ ;\ S \mid \textbf{if}\ (*)\ \{S\}\ \textbf{else}\ \{S\} \mid \textbf{while}\ (A)\ \textbf{inv}\ A\ \{S\} \mid \vec{V} := m(\vec{V})$$
$$\mid \textbf{inhale}\ A \mid \textbf{exhale}\ A \mid \textbf{assume}\ A \mid \textbf{var}\ \vec{V} \mid \textbf{havoc}\ \vec{V} \mid \textbf{custom}\ O \mid \textbf{skip}$$

$S\ ;\ S$ is a sequential composition, **if** $(*)$ $\{S\}$ **else** $\{S\}$ a non-deterministic conditional branching, **while** (b) **inv** I $\{s\}$ a loop, $\vec{V} := m(\vec{V})$ a method call, **assume** assumes an assertion, **var** declares a list of variables while **havoc** havocs a list of variables, **custom** allows one to encode statements not presented here, and **skip** does nothing.

Remark 5.2 *We already wrote about* **`inhale`** *and* **`exhale`** *in Viper. The* **inhale** *and* **exhale** *statements presented here are different and behave differently. We simply require them to be equivalent when applied to annotations (eg.* **`acc(x.f) && x.f == 5`***), but they can behave differently with other assertions.*

When injecting Viper into this framework, we therefore distinguish **`inhale`** *and* **`exhale`** *which come from loop and method call verification with the ones which do not (which should be encoded with* **custom***).*

Most of these statements are given the usual semantics, and their semantics are described formally throughout this chapter. We use a non-deterministic if in this framework, since it is simpler to define and more general than a deterministic if, which we define as follows:

Definition 5.3 ***Deterministic Ifs***

$$\begin{aligned}\textbf{if } (b)\ \{s_1\}\ \textbf{else}\ \ \{s_2\}\ &:= \textbf{if } (*)\ \{\textbf{assume } b\ ;\ s_2\}\ \textbf{else}\ \{\textbf{assume } \neg b\ ;\ s_2\}\\ \textbf{if } (b)\ \{s_1\}\ &:= \textbf{if } (*)\ \{\textbf{assume } b\ ;\ s\}\ \textbf{else}\ \{\textbf{assume } \neg b\}\end{aligned}$$

Moreover, the statements for declaring and havocing variables, as well as the statements for calling methods, are defined with lists of variables instead of single variables, to allow more expressivity.

5.1.3 Semantics

We define as follows the semantics with respect to a program:

Definition 5.4 ***Method, program and semantics***

- *A **method** is a quadruple $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s)$ where m is a string (name of the method), $\overrightarrow{args}$ and $\overrightarrow{rets}$ are lists of variable names (representing arguments and return variables), P (precondition) and Q (postcondition) are semantic assertions (functions from states Σ to booleans), and s is a statement (the method body).*
- *A **program** Pr is a finite set of methods.*
- *We denote $semantics_{Pr}(\varphi, s)$ the semantics of the statement s executed with the initial state φ, with respect to the program Pr. It is either an error or a set of states.*[1]

We consider a fixed *semantics* function in this section, which we characterize throughout this chapter. Finally, we define two auxiliary functions based on the *semantics* function:

Definition 5.5 *Let Pr be a program, A be a set of states and s be a statement. We define two functions **ver** and **sem** such that*

$$\begin{aligned}ver_{Pr}(A, s) &:= (\forall \varphi \in A. semantics_{Pr}(\varphi, s) \neq Error)\\ sem_{Pr}(A, s) &:= \cup_{\varphi \in A} semantics_{Pr}(\varphi, s)\end{aligned}$$

and $sem_{Pr}(A, s)$ is defined if and only if $ver_{Pr}(A, s)$.

[1] The set of states models the non-determinism of the semantics, while the error signifies a failure (and ignores states which may not fail).

$ver_{Pr}(A, s)$ is true if and only if the statement s verifies with all states from the set A, with respect to the program Pr. In this case, $sem_{Pr}(A, s)$ is the set of all states resulting from executing s from all states of A. In this chapter, we mostly characterize the semantics using these notations. The original *semantics* function can be fully reconstructed from these two auxiliary functions.

5.2 Store of a state

In order to define the semantics of **var** and **havoc**, we need to introduce the concept of store of a state. We first explain the requirements on the store function σ for a state, and then define the semantics of **var** and **havoc**.

5.2.1 Store function

In Viper, the store of a state can be seen as a finite mapping from variable names to values of the right type. In this framework, we do not need as much precision and information. A state in our framework may still use a mapping to define a store. However, for the purpose of our framework (proving properties on static inlining), we only require a projection of this mapping to the sets of variables it defines. The validity of assertions such as $x == 5$ is handled by the separation algebra.

Requirement 5.6 *The **store function** σ from states to sets of variable names must satisfy the following rules (for all states φ, a, b, and all variable names x):*

1. $\sigma(u) = \varnothing$
2. $x \in \sigma(\varphi) \Longrightarrow (\exists c \in \Sigma. \sigma(c) = \{x\} \wedge c << \varphi)$
3. $a\#b \Longrightarrow \sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$
4. $\sigma(\varphi) = \sigma(|\varphi|)$
5. $a\#b \wedge a \text{ is pure} \wedge \sigma(a) \subseteq \sigma(b) \Longrightarrow a << b$
6. $\sigma(a) \cap \sigma(b) = \varnothing \wedge a \text{ is pure} \Longrightarrow a\#b$

As an example, if a state φ has a store which maps x to 5, y to 7, and z to 42, then $\sigma(\varphi) = \{x, y, z\}$. The first three axioms deal with combining states. The empty state has an empty store (axiom 1). It is always possible to find a weaker state than φ with a store containing only one variable x (which is defined in the store of φ) (axiom 2). The store of the addition of two states is the union of their stores (axiom 3). Axioms 4 and 5 formalize that the core of a state represents exactly its store. The store of a state is the store of its core (axiom 4). If a and b can be added, this means that they agree on the values of local variables which both stores define. Therefore, if a is pure (only a store) and its store is included in the store of b, then a is weaker than b

(axiom 5). Finally, it is always possible to add new variables to a state (axiom 6).

Using the *finiteness* axiom of the separation algebra, we can prove the following lemma:

Lemma 5.7 $\forall \varphi \in \Sigma.\sigma(\varphi)$ *is finite.*

5.2.2 Var and havoc

Declaring variables verifies if and only if no variable from this list is already declared, whereas havocing variables verifies if and only if all variables from this list are already declared. For readability purposes, and throughout this thesis, we consider implicit conversions between lists of variable names and sets of variable names.

Definition 5.8 ***Verification of* var *and* havoc***:*

$$ver_{Pr}(\{\varphi\}, \textbf{var } \vec{l}) \iff \vec{l} \cap \sigma(\varphi) = \varnothing$$
$$ver_{Pr}(\{\varphi\}, \textbf{havoc } \vec{l}) \iff \vec{l} \subseteq \sigma(\varphi)$$

Remark 5.9 *This definition of* $ver_{Pr}(\{\varphi\}, \textbf{var } \vec{l})$ *actually makes* **var** *not* ***safeMono****. An alternative definition where* **var** *defines variables which are not already defined and leaves the other variables unchanged would make all* **var** *statements* ***safeMono****. This alternative definition can also be used with the custom interface. If it can be externally justified that no variable names collide, then the two statements are equivalent in terms of semantics.*

The semantics of **var** and **havoc**, when they verify, are a bit different. **var** has to define (add to the store) the new variables, with all possible values. **havoc** has to remove the variables, and then define them again with all possible values. To define these two semantics, we define two auxiliary functions: One to define variables with all possible values, and one to remove variables from a state. We define below the former:

Definition 5.10 *h* ***function****:*

$$h(\vec{l}) = \{\varphi | \varphi \text{ is pure} \wedge \sigma(\varphi) = \vec{l}\}$$

$h(\vec{l})$ is the set of all states with no impure resources, only a store, where the variables which are defined are exactly $\vec{l}$. This function is sufficient to define the semantics of **var** when it verifies:

Definition 5.11 ***Semantics of* var***:*

$$sem_{Pr}(\{\varphi\}, \textbf{var } \vec{l}) = \{\varphi\} \oplus h(\vec{l})$$

This is the definition we want. Indeed, adding this set to the singleton $\{\varphi\}$ results in all states which correspond to φ plus the definition of the variables contained in $\vec{l}$, with all possible values.

To define the semantics of **havoc**, we need a function which "undefines" (removes) variables from a state. Such a function exists because of the following lemma:

Lemma 5.12

$$\forall \varphi \in \Sigma. \exists! p \in \Sigma. p << \varphi \wedge \sigma(p) = \sigma(\varphi) - V \wedge p \textit{ is pure}$$

This lemma shows that there exists a unique store (pure state) which defines the variables in $\sigma(\varphi) - \vec{l}$ and matches exactly the store of φ excluding $\vec{l}$. It is now easy to define a function which only removes the part of the store corresponding to $\vec{l}$:

Definition 5.13 $\overline{h}$ ***function:***
Let p be the unique p such that $p << \varphi \wedge \sigma(p) = \sigma(\varphi) - \vec{l} \wedge p$ is pure. Then

$$\overline{h}(\varphi, \vec{l}) = p \oplus C(\varphi)$$

We can finally define the semantics of **havoc**.

Definition 5.14 **havoc** ***semantics:***

$$sem_{Pr}(\{\varphi\}, \textbf{havoc } \vec{l}) = \{\overline{h}(\varphi, \vec{l})\} \oplus h(\vec{l})$$

This is the semantics described above: We remove $\vec{l}$ from the store of φ, and then we add again the variables from $\vec{l}$, but with all possible values.

5.3 Assertions

The purpose of this section is to define the semantics of **inhale**, **exhale** and **assume**. We first explain how to encode syntactic assertions from Viper into semantic assertions in our separation algebra. We then define two properties, *supported* and *intuitionistic*, which are satisfied by semantic assertions encoded from Viper annotations. Afterwards, we discuss a mismatch between syntactic and semantic assertions, how this gap could be bridged, and how we currently define the verification of **inhale**, **exhale** and **assume** in this framework. This means that we only need to model the behavior of **`inhale`** and **`exhale`** on this restricted class of semantic assertions. Eventually, we define the concept of the minimal satisfying set of an assertion, and use this concept to express the semantics of **inhale** and **exhale**.

5.3.1 From syntactic to semantic assertions

Viper assertions written in Viper programs are syntactic assertions. However, we do not want to define a syntax for assertions in our framework. We therefore encode syntactic assertions into semantic assertions. Following [21, 25, 5, 18], we define semantic assertions as follows:

Definition 5.15 *A **semantic assertion** on a set of states Σ is a function $\Sigma \to Bool$.*

Remark 5.16 *It is equivalent to see semantic assertions as elements of $\mathcal{P}(\Sigma)$.*

Semantic assertions aim at capturing syntactic assertions (as in Viper). For an infinite number of states, there is an uncountable number of semantic assertions (Cantor's theorem), whereas there is only a countable number of syntactic assertions (defined on a finite alphabet). Therefore, our hope is that every syntactic assertion can be represented as a semantic assertion, and thus that working with semantic assertions is more general than working with syntactic assertions. However, Section 5.3.3 argues that there is a mismatch (at least for assertions in Viper).

Definition 5.15 is sufficient to define the semantics of **assume** when the statement verifies (the verification of **assume** statements is defined in Section 5.3.3).

Definition 5.17 ***Partial semantics of assume***

$$sem_{Pr}(\{\varphi\}, \textbf{assume } P) = \begin{cases} \{\varphi\} & \text{if } P(\varphi) \\ \varnothing & \text{otherwise} \end{cases}$$

assume statements simply filter states which do not satisfy the assertion. That is, they stop the execution of these states.

5.3.2 Annotations are supported and intuitionistic

Semantic assertions are too general for our purpose, since we just want to match the behavior of **`inhale`** and **`exhale`** on annotations, and annotations are syntactic assertions with restrictions. Indeed, annotations have to be self-framing and are not allowed to use permission introspection. We can relate this restriction with two notions for semantic assertions.

As explained in Chapter 1, not using permission introspection introduces a certain monotonicity, which can be related to the notion of intuitionistic assertions [21] :

Definition 5.18 *A semantic assertion P is **intuitionistic** if and only if*

$$(\forall \varphi, \varphi' \in \Sigma. \varphi << \varphi' \implies (P(\varphi) \Rightarrow P(\varphi')))$$

This means that if a state satisfies an intuitionistic assertion, then all stronger states satisfy it too.

Moreover, syntactic self-framing assertions from Viper are encoded into semantic supported assertions (more explanations are given in Chapter 8). As explained in Section 2.2.1, an assertion is self-framing if it includes permissions to at least the locations it reads. This property, combined with other restrictions on assertions in Viper, implies that its representation as a semantic assertion is supported [18]:

Definition 5.19 *An assertion P is **supported** if and only if*

$$(\forall \varphi \in \Sigma.P(\varphi) \\ \implies \exists \varphi_m \in \Sigma.\varphi_m << \varphi \wedge P(\varphi_m) \wedge (\forall \varphi' \in \Sigma.\varphi' << \varphi \wedge P(\varphi') \Rightarrow \varphi_m << \varphi'))$$

In other words, a semantic assertion P is supported if and only if, for any state φ that satisfies P, it is possible to find a support φ_m weaker than φ, which also satisfies P. Being a support means that all states weaker than φ satisfying P are stronger than the support φ_m.

In Viper, a state needs at least some (> 0) permission to a location to have information about it. As an example, the assertion `x.f == 5` (not self-framing) is not a supported assertion, since it is impossible to find a support for this assertion and a state satisfying this assertion. Indeed, any state satisfying this assertion has to have some permission to $x.f$, and so does its support if it would exist. However, any state satisfying $x.f == 5$ with less permission (but not zero) than the support would be weaker than the support while satisfying the assertion. Such a state exists: It is for example sufficient to divide the permission by two. This contradicts the definition of the support.

Another example of a not supported assertion is **acc**`(x.f) || `**acc**`(y.f)` (such an assertion is not allowed in Viper). Take the state φ which has full permission to both $x.f$ and $y.f$. φ satisfies the assertion. Take then the state φ_x (resp. φ_y) with only full permission to $x.f$ (resp. $y.f$). φ_x and φ_y satisfy the assertion and are weaker than φ. φ_x and φ_y cannot be the support, since φ_x is not weaker than φ_y, and vice-versa. Moreover, a state weaker than both φ_x and φ_y (as a support for φ should be) would have no permission to $x.f$ and $y.f$, therefore not satisfying the assertion.

Semantic assertions encoded from Viper annotations are *well-formed*:

Definition 5.20 ***Well-formed assertions***
*A semantic assertion P is **well-formed**, written $wfAssert(P)$, if and only if P is **supported** and **intuitionistic**.*

Remark 5.21 *This definition of **well-formed** semantic assertions is a bit too restrictive. Some Viper annotations, such as disjunctions, do not correspond to **supported***

assertions. To prove the soundness theorem (Chapter 6), a semantic assertion P used as an annotation needs only to satisfy the following property, which is satisfied by all ***well-formed*** *assertions:*

$$\forall \varphi \in \Sigma.P(\varphi) \iff \{\varphi\} >> Inh(P)$$

The definition of ***well-formed*** *assertions will be modified with this property in the future.*

In the next subsections, we model the behavior of **inhale** and **exhale** based on supported and intuitionistic semantic assertions. Chapter 8 explains why Viper annotations (syntactic assertions) are encoded into supported and intuitionistic semantic assertions, which justifies this choice.

5.3.3 Well-defined assertions: A mismatch between syntactic and semantic assertions

assume statements in Viper do not always verify. They verify if and only if they are well-defined in a state. As an example, `assume x >= 7 && y <= 9` verifies only in a state where $\vec{x}$ and $\vec{y}$ are defined. We want to transpose this concept of well-definedness from syntactic assertions to semantic assertions, in order to describe when **assume**, **inhale** and **exhale** verify. This is encoded via the *wellDefinedAssert* function from the assertions interface. The difficulty boils down to the definition of such a property, and by extension to the encoding of a syntactic assertion into a semantic assertion.

It does not actually work, because of a mismatch between syntactic and semantic assertions. To illustrate this mismatch, consider two Viper assertions P and Q, where $P := (x.f == x.f)$, and $Q := (a == a)$. P is well-defined with respect to a state if and only if this state has at least some permission to $x.f$, and Q is well-defined with respect to a state if and only if a is defined in this state. P and Q are always true when they verify. If we decide to define P and Q as being always true, then we cannot distinguish them from the semantic assertion *true*, which is well-defined in all states. If we decide to define P and Q to be true when they verify and false otherwise, then P is indistinguishable from the assertion $perm(x.f) > 0$ (well-defined in all states) and Q would be indistinguishable from an assertion such as $defined(a)$. Such an assertion ($defined(a)$) does not exist in Viper, but it could exist. In this case, it would always be well-defined, and would be true in states where a is defined in the store.

These two examples (P and Q) show the impossibility to distinguish different syntactic assertions when they are encoded into semantic assertions. This is an issue we want to tackle in the future, by using extended semantic assertions, which would be functions from states to some boolean (well-defined) or some special value signifying that the semantic assertion is not well-defined.

In our current model, to express when a semantic assertion is well-defined, we first need define the set of variables read by an assertion:

Definition 5.22 ***Set of variables read by an assertion***

$$readAssert(P) := \cup_{i \in Inh(P)} \sigma(i)$$

This function captures the set of variables which have an effect in determining the value of an assertion. Intuitively, it corresponds to the minimum set of variables needed to syntactically define the assertion. We now define the three requirements we impose on the *wellDefinedAssert* function for semantic assertions as follows:

Requirement 5.23 ***Well-defined assertion***
wellDefinedAssert is a function which takes an assertion and a state as input, and returns a boolean. This function must satisfy the following properties, for all assertions P and all states φ and φ':

1. $wfAssert(P) \implies (wellDefinedAssert(P, \varphi) \Leftrightarrow readAssert(P) \subseteq \sigma(\varphi))$
2. $wellDefinedAssert(P, \varphi) \iff wellDefinedAssert(\neg P, \varphi)$
3. $P(\varphi) \implies wellDefinedAssert(P, \varphi)$

The first requirement says that, for well-formed assertions (that is supported and intuitionistic assertions, in particular annotations), this assertion is well-defined in a state if and only if the state defines all the variables it *reads*. The intuition is that well-formed semantic assertions represent self-framing syntactic assertions, and self-framing assertions cannot fail, by definition, because of permission. The issue has therefore to come from variables not defined. The second requirement simply follows a syntactic logic, in which if an assertion is well-defined, then its negation is also well-defined. The third requirement states that if a state satisfies an assertion, then this assertion is well-defined in this state. This interface and these requirements should be modified in the future with an extended form of semantic assertions.

We can now express the verification of **inhale**, **exhale**, and **assume** statements:

Definition 5.24 ***Verification of inhaling, exhaling, and assuming***

$$\begin{aligned} ver_{Pr}(\{\varphi\}, \textbf{inhale } P) &\iff wellDefinedAssert(P, \varphi) \\ ver_{Pr}(\{\varphi\}, \textbf{exhale } P) &\iff wellDefinedAssert(P, \varphi) \wedge P(\varphi) \\ ver_{Pr}(\{\varphi\}, \textbf{assume } P) &\iff wellDefinedAssert(P, \varphi) \end{aligned}$$

That is, these statements verify if and only if they are well-defined in the current state.

5.3.4 Minimal satisfying set of an assertion

We now introduce another concept before defining the semantics of **inhale** and **exhale**. As explained in Section 4.2, inhaling an assertion adds permission from accessibility predicates and assumes pure information. In the case of adding permission, this works similarly to adding a state to the initial state. In the case of assuming an assertion talking about pure resources, the execution is stopped if this assertion is not satisfied, and this is modeled using the partial definedness of the addition of our separation algebra. Inhaling, when verifying, corresponds to adding a set of states:

Definition 5.25 *Let P be an assertion. The **minimal satisfying set** of P is defined as*

$$Inh(P) := \{\varphi \in \Sigma | P(\varphi) \wedge (\forall \varphi' \in \Sigma. \varphi' << \varphi \wedge \varphi' \neq \varphi \Rightarrow \neg P(\varphi'))\}$$

$Inh(P)$ is the set of minima of the states satisfying P. Using this set, we define the semantics (when it verifies) of **inhale** as follows:

Definition 5.26 ***Semantics of* inhale***:*

$$sem_{Pr}(\{\varphi\}, \textbf{inhale } P) := \{\varphi\} \oplus Inh(P)$$

This works indeed as expected for modeling Viper. The addition of states adds the permissions. As an example, `inhale acc(x.f, 1/2)` adds half permission to the heap location $x.f$. If this new permission is strictly greater than 1, this addition is not defined, it therefore results in an empty set, which models *magic* states in Viper. The same thing happen with assertions which deal with values, such as $x == 5$ or $y.f == 7$. These assertions act as filters: It they are satisfied by a state, then the addition is defined. However, if they are not satisfied, the addition is not defined, and thus also results in an empty state.

Example 5.27 *Let $P :=$ `acc(x.f, 1/2) && x.f == 5`. From the definition, $Inh(P)$ is the set of all states whose stores only define x (a reference), have half permission to $x.f$, and where $x.f = 5$.*

- *If φ has no permission to $x.f$, then $\{\varphi\} \oplus Inh(P)$ results in a set with only one state, this state being the sum of φ and the unique state of $Inh(P)$ which defines the same x as φ. Therefore, the resulting state has half permission to $x.f$, and $x.f = 5$.*
- *If φ has some permission to $x.f$, then it also carries a value for $x.f$.*
  - *If this value is not 5, then the sum of sets becomes the empty set (magic), since no element of $Inh(P)$ is compatible with φ.*

  - *If this value is* 5*, then we have almost the same situation as in the first point, only in this case the resulting permission is* $\frac{1}{2}$ *more than before.*

- *Finally, if* φ *has more than half permission to* $x.f$*, then* φ *is not compatible with any element from* $Inh(P)$*, resulting once again in an empty set (magic).*

As shown in Section 4.2, an **exhale** statement in Viper removes the permissions added by an **inhale**, but keeps the pure part. We define the semantics of **exhale** as follows:

Definition 5.28 *Semantics of* **exhale**

$$sem_{Pr}(\{\varphi\}, \textbf{exhale } P) := \{|i| \oplus r | i \in Inh(P) \land r \in \Sigma \land \varphi = i \oplus r\}$$

It is not clear from the definition that this matches **exhale** in Viper. It actually does not have the same behavior in general, but it does in a special case, when an assertion is supported and intuitionistic. The situation is similar with **inhale**, which matches **inhale** in Viper in the case of an assertion supported and intuitionistic (*well-formed*).

Recall that self-framing assertions in Viper are encoded into semantic assertions which are both supported and intuitionistic, thus well-formed. The next lemma shows why this property is sufficient to have a correspondence between Viper and this framework when dealing with annotations.

Lemma 5.29

$$wfAssert(P) \implies (P(\varphi) \Leftrightarrow \exists! i \in Inh(P).i << \varphi)$$

Proof Assume $wfAssert(P)$, that is P is intuitionistic and supported.

$\exists! i \in Inh(P).i << \varphi \implies P(\varphi)$ follows from the definition of intuitionistic and the definition of $Inh(P)$.

$P(\varphi) \implies \exists! i \in Inh(P).i << \varphi$ follows from the definition of supported. □

Using this lemma, we can now formalize the motivation from Section 4.2. That is, exhale verifies after inhale, and inhaling and exhaling the same assertion only deals with pure resources.

Lemma 5.30 *Assume* $wfAssert(P)$ *and* $ver_{Pr}(\{\varphi\}, \textbf{inhale } P)$*. Then*

1. $ver_{Pr}(\{\varphi\}, \textbf{inhale } P \,;\, \textbf{exhale } P)$
2. $sem_{Pr}(\{\varphi\}, \textbf{inhale } P \,;\, \textbf{exhale } P) = \{\varphi\} \oplus \{|i| | i \in Inh(P)\}$

Proof We have $sem_{Pr}(\{\varphi\}, \textbf{inhale } P) = \{\varphi\} \oplus Inh(P)$. Let $\varphi' \in \{\varphi\} \oplus Inh(P)$, then $\exists i \in Inh(P).\varphi' = \varphi \oplus i$. Since P is intuitionistic, we have $P(\varphi')$, thus we have $wellDefinedAssert(P, \varphi')$ from Requirement 5.23. Therefore, $ver_{Pr}(\{\varphi'\}, \textbf{exhale } P)$, hence $ver_{Pr}(\{\varphi\}, \textbf{inhale } P \,;\, \textbf{exhale } P)$.

Moreover, using Lemma 5.29, we have that there is a unique $i \in Inh(P)$ such that $i << \varphi'$. Hence,

$$\begin{aligned} sem_{Pr}(\{\varphi'\}, \textbf{exhale } P) &= \{|i'| \oplus r | i' \in Inh(P) \wedge r \in \Sigma \wedge \varphi' = i' \oplus r\} \\ &= \{|i| \oplus \varphi\} \\ &= \{\varphi\} \oplus \{|i|\} \end{aligned}$$

It follows that

$$sem_{Pr}(\{\varphi\}, \textbf{inhale } P \textbf{ ; exhale } P) = \{\varphi\} \oplus \{|i| | i \in Inh(P)\} \qquad \square$$

Finally, the correspondence between the semantics of **`inhale`** and **`exhale`** statements in Viper and the semantics of **inhale** and **exhale** in our language does not work in general when assertions are not annotations.

Example 5.31 *Let $P := acc(x.f, perm(x.f))$. P is well-defined in all states. In Viper, **`inhale`** `P` corresponds to doubling the permission held to $x.f$, and **`exhale`** `P` removes all the permission to $x.f$. In our framework, on the other hand, P is always true and well-defined, it therefore always verifies and* **inhale** *P and* **exhale** *P do nothing.*

5.4 While loops

We define the semantics of while loops in this section. As explained in Section 2.2.4, the semantics of a loop involves havocing the variables modified by the body of the loop. We therefore first define which variables are modified by a statement, and then define the semantics of while loops.

Definition 5.32 ***Variables modified by a statement***

$$\begin{aligned} & modif(s_1 \textbf{ ; } s_2) = modif(s_1) \cup modif(s_2) \\ | \ & modif(\textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) = modif(s_1) \cup modif(s_2) \\ | \ & modif(\vec{y} := m(\vec{x})) = y \\ | \ & modif(\textbf{while } (b) \textbf{ inv } I \ \{s\}) = modif(s) \\ | \ & modif(\textbf{assume } P) = \varnothing \\ | \ & modif(\textbf{inhale } P) = \varnothing \\ | \ & modif(\textbf{exhale } P) = \varnothing \\ | \ & modif(\textbf{var } \vec{l}) = \vec{l} \\ | \ & modif(\textbf{havoc } \vec{l}) = \vec{l} \\ | \ & modif(\textbf{skip}) = \varnothing \\ | \ & modif(\textbf{custom } o) = modifCustom(o) \end{aligned}$$

Using this function, we define the semantics of while loops as follows:

Definition 5.33 *Semantics of while loops*
Let $\vec{l} := modif(s) \cap \sigma(\varphi)$. *Then*

$$\begin{aligned} & ver_{Pr}(\{\varphi\}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) \\ \Longleftrightarrow & ver_{Pr}(\{|\varphi|\}, \textbf{havoc } \vec{l} \ ; \textbf{inhale } I \ ; \textbf{assume } b \ ; s \ ; \textbf{exhale } I) \\ \wedge & ver_{Pr}(\{\varphi\}, \textbf{exhale } I \ ; \textbf{havoc } \vec{l} \ ; \textbf{inhale } I \ ; \textbf{assume } \neg b) \end{aligned}$$

and

$$\begin{aligned} & sem_{Pr}(\{\varphi\}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) \\ := & sem_{Pr}(\{\varphi\}, \textbf{exhale } I \ ; \textbf{havoc } \vec{l} \ ; \textbf{inhale } I \ ; \textbf{assume } \neg b) \end{aligned}$$

The first part of the loop verification verifies if the invariant is an actual invariant. We begin with a state with no impure resources (the core), and then forget (havoc) the values of variables which can be modified by the loop. We then inhale the invariant and assume that we are in an iteration of the loop (b), execute the loop body s, and then make sure the state still satisfies the invariant. If this verifies, the possible states after the loop, represented by $sem_{Pr}(\varphi, \textbf{while } (b) \textbf{ inv } I \ \{s\})$, can be computed by exhaling the loop invariant, forgetting the values of the variables modified by the loop, and then inhaling the invariant as well as assuming we are out of the loop ($\neg b$).

Remark 5.34 *This definition defines* $\vec{l}$ *as* $modif(s) \cap \sigma(\varphi)$, *but the Isabelle formalization (Appendix A) defines it as* $modif(s)$ *only. The Isabelle formalization should be modified with the definition presented here. Indeed, in the case where variables are declared inside the loop body, using* $\vec{l} := modif(s)$, *any verification of this loop would fail. The variables would have both to be defined before the loop (to be havoced), and not to be defined inside the loop body (where they are declared).*

5.5 Renaming interface

We have now defined the semantics for while loops, and want to define the semantics of method calls. When a method is called, the variables in the precondition and the postcondition of the method must be renamed. If the method is $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s)$, and the method call is $\vec{y} := m(\vec{x})$, we need to rename the variables of the precondition P, which only speaks about $\overrightarrow{args}$. We also need to rename the variables of the postcondition Q, which only speaks about $\overrightarrow{args}$ and $\overrightarrow{rets}$. $\overrightarrow{args}$ should be renamed to $\vec{x}$, and $\overrightarrow{rets}$ to $\vec{y}$.

On top of this, inlining a method call requires renaming the method body. As we explain in Chapter 6, we need to make sure variables declared in

the method body do not interfere with variables already defined elsewhere, to avoid variable capturing. This is the reason why we require a powerful renaming interface, which is able to both rename $\overrightarrow{args}$ to $\vec{x}$ and $\overrightarrow{rets}$ to $\vec{y}$, but which can also avoid a list of variables already defined elsewhere.

The renaming functions we present in this section are built on a single renaming function, which renames one element (a variable name) to another one. This renaming function, on top of the element to rename, takes as a parameter a renaming quadruple:

Definition 5.35 *A **renaming quadruple** is a quadruple of lists of variables*

$$(\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain})$$

where

- $\overrightarrow{old}$ *is a list of variables which should be renamed to variables from* $\overrightarrow{new}$.
- $\overrightarrow{avoid}$ *is a list of variables to avoid.*
- $\overrightarrow{domain}$ *is the list of variables on which this renaming function should be applied.*

The idea of this definition is as follows. When calling a method, we need to rename its precondition and postcondition. As an example, take the method $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s)$, and take the method call $\vec{y} := m(\vec{x})$. We want to rename P and Q, which speak about $\overrightarrow{args}$ and $\overrightarrow{rets}$, to speak about $\vec{x}$ and $\vec{y}$. In this case, $\overrightarrow{old}$ would be the concatenation of $\overrightarrow{args}$ and $\overrightarrow{rets}$, where $\overrightarrow{new}$ would be the concatenation of $\vec{x}$ and $\vec{y}$.

$\overrightarrow{avoid}$ and $\overrightarrow{domain}$ are useful for inlining. When we inline a method call, we want to rename variables declared in the method body with names of variables which are not defined in the rest of the program, to avoid interferences with the calling context (i.e., to avoid variable capturing). $\overrightarrow{avoid}$ is the list of variables defined elsewhere which should be renamed to something else. Finally, we want to be able to invert the renaming, to easily show properties about renamed statements. This is why we need a $\overrightarrow{domain}$ list. All variables in $\overrightarrow{domain}$ are renamed to variables which are not in $\overrightarrow{avoid}$. However, to make this renaming invertible, we need to rename some variables to variables in $\overrightarrow{avoid}$, to make the renaming surjective. We therefore get the *avoid* property only on $\overrightarrow{domain}$, thus $\overrightarrow{domain}$ should represent the variables which are renamed.

To get these properties, we first impose some conditions on a renaming quadruple:

Definition 5.36 *A **renaming quadruple** $(\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain})$ is **well-formed**, denoted by $wfRenaming((\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain}))$, if and only if*

1. $length(\overrightarrow{old}) = length(\overrightarrow{new})$.
2. $distinct(\overrightarrow{old})$ *(elements of* $\overrightarrow{old}$ *are all distinct).*
3. $distinct(\overrightarrow{new})$ *(elements of* $\overrightarrow{new}$ *are all distinct).*

We formally express the required properties as follows:

Requirement 5.37 ***Rename an element, invert a renaming quadruple***
Let $t := (\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain})$*, and assume* $wfRenaming(t)$*. Then*

1. $wfRenaming(renameInv(t))$
2. $rename(renameInv(t), rename(x, t)) = x$
3. $\forall i < length(\overrightarrow{old}).rename(t, \overrightarrow{old}[i]) = \overrightarrow{new}[i]$
4. $x \in \overrightarrow{domain} - \overrightarrow{old} \implies rename(t, x) \notin \overrightarrow{avoid}$

The first one means that the inverse of a renaming quadruple is a well-formed renaming quadruple. The second one means that the inverse of a renaming quadruple inverts the renaming. The third one says that the renaming function actually renames $\overrightarrow{old}$ variables to $\overrightarrow{new}$ variables. Finally, the last one shows that variables from the $\overrightarrow{domain}$ are renamed in something which is not in $\overrightarrow{avoid}$. These are the core functions and properties we need.

We now impose requirements on how to rename states, leveraging the properties of *rename*:

Requirement 5.38 *Assume* $wfRenaming(t)$*. Then*

1. $\sigma(renameState(t, \varphi)) = \{rename(t, x) | x \in \sigma(\varphi)\}$
2. $renameState(t, \varphi_1 \oplus \varphi_2) = renameState(t, \varphi_1) \oplus renameState(t, \varphi_2)$
3. $(\forall x \in \sigma(\varphi).rename(t, x) = x) \implies renameState(t, \varphi) = \varphi$
4. $wfRenaming(t_1) \wedge wfRenaming(t_2) \wedge$
   $(\forall x \in \sigma(\varphi).rename(t, x) = rename(t_2, rename(t_1, x)))$
   $\implies renameState(t, \varphi) = renameState(t_2, (renameState(t_1, \varphi)))$

These requirements follow a simple definition of renaming states, which just renames the variables defined in the stores. The first requirement is about the store of a renamed state. The second one shows that renaming a sum of states is equivalent to adding the renamed states. The third one says that, if all variables are left unchanged by the renaming function, then the state is left unchanged. Finally, the last one says that if the composition of two renaming functions is the same as one on the variables defined by φ, then it is also the same at the state level.

We now define how to rename semantic assertions, based on renaming states:

Definition 5.39 ***Rename assertions***

$$renameA(t, P)(\varphi) := P(renameState(renameInv(t), \varphi))$$

Renaming a semantic assertion is the same as applying it to the state renamed in the inverse direction. As an example, take the state φ_y which only defines y and the assertion $P := (x == 5)$. Let $t := ([x], [y], [], [])$. Renaming the assertion with t gives $y == 5$, which is the same as applying P to φ_y renamed with the invert of t: In this case, $renameState(renameInv(t), \varphi_y)$ only defines x.

We impose two more requirements for renaming assertions:

Requirement 5.40 *Assume* $wfRenaming(t)$. *Then*

1. $wellDefinedAssert(P, \varphi)$
   $\implies wellDefinedAssert(renameA(t, P), renameState(t, \varphi))$

2. $wfAssert(P, \varphi) \implies wfAssert(renameA(t, P), renameState(t, \varphi))$

That is, renaming an assertion keeps the assertion well-defined with respect to a state (but renamed), and well-formed (namely *supported* and *intuitionistic*).

We leverage these renaming functions for elements, states and assertions to define a function to rename statements. We first define how to rename a list of variable names:

Definition 5.41 ***Rename a list of variable names***

$$\forall i < length(\vec{l}).renameList(t, \vec{l})[i] = rename(t, \vec{l}[i])$$

This simply means renaming every element of the list. We can now define the *renameS* function to rename statements:

Definition 5.42 ***Rename a statement***

$$
\begin{aligned}
&\mathit{renameS}(t, \textbf{inhale } P) = \textbf{inhale } \mathit{renameA}(t, P)\\
&\mid \mathit{renameS}(t, \textbf{exhale } P) = \textbf{exhale } \mathit{renameA}(t, P)\\
&\mid \mathit{renameS}(t, s_1 \,;\, s_2) = \mathit{renameS}(t, s_1) \,;\, \mathit{renameS}(t, s_2)\\
&\mid \mathit{renameS}(t, \textbf{var } \vec{l}) = \textbf{var } \mathit{renameList}(t, \vec{l})\\
&\mid \mathit{renameS}(t, \textbf{havoc } \vec{l}) = \textbf{havoc } \mathit{renameList}(t, \vec{l})\\
&\mid \mathit{renameS}(t, \textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) = \textbf{if } (*) \ \{\mathit{renameS}(t, s_1)\} \textbf{ else } \{\mathit{renameS}(t, s_2)\}\\
&\mid \mathit{renameS}(t, \textbf{while } (b) \textbf{ inv } I \ \{s\}) =\\
&\quad \textbf{while } (\mathit{renameA}(t, b)) \textbf{ inv } \mathit{renameA}(t, I) \ \{\mathit{renameS}(t, s)\}\\
&\mid \mathit{renameS}(t, \vec{y} := m(\vec{x})) = \mathit{renameList}(t, \vec{y}) := m(\mathit{renameList}(t, \vec{x}))\\
&\mid \mathit{renameS}(t, \textbf{assume } b) = \textbf{assume } \mathit{renameA}(t, b)\\
&\mid \mathit{renameS}(t, \textbf{custom } o) = \textbf{custom } \mathit{renameCustom}(t, o)\\
&\mid \mathit{renameS}(t, \textbf{skip}) = \textbf{skip}
\end{aligned}
$$

This renaming function is useful for defining the inlining function in Chapter 6. Requirements on *renameCustom* are given in Section 5.7.

5.6 Method calls

Using our function to rename semantic assertions, we can now define the semantics of method calls:

Definition 5.43 ***Semantics of method calls***
Assume $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s) \in Pr$.
Let $t := (\overrightarrow{args} ++ \overrightarrow{rets}, \vec{x} ++ \vec{y}, \varnothing, \varnothing)$ *(where* $++$ *denotes concatenation),* $P' := \mathit{renameA}(t, P)$ *and* $Q' := \mathit{renameA}(t, Q)$. *Then*

$$
\begin{aligned}
&\mathit{ver}_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x}))\\
\Longleftrightarrow &\mathit{ver}_{Pr}(\{\varphi\}, \textbf{exhale } P' \,;\, \textbf{havoc } \vec{y} \,;\, \textbf{inhale } Q') \wedge \vec{x} \cup \vec{y} \subseteq \sigma(\varphi)
\end{aligned}
$$

and

$$
\mathit{sem}_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x})) := \mathit{sem}_{Pr}(\{\varphi\}, \textbf{exhale } P' \,;\, \textbf{havoc } \vec{y} \,;\, \textbf{inhale } Q')
$$

To verify, the state φ needs to at least define the variables from $\vec{x}$ and $\vec{y}$. The verification and behavior of the method call is then equivalent to exhaling the renamed precondition, havocing the return variables, and inhaling the renamed postcondition.

5.7 Rest of the semantics

We now proceed to complete the definition of the semantics, by first defining the semantics of control structures, and then defining the requirements on *custom* statements.

5.7.1 Control structures

A state enters both branches of a non-deterministic if, hence the following semantics:

Definition 5.44 ***Semantics of non-deterministic if***

$$ver_{Pr}(\{\varphi\}, \textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) \iff ver_{Pr}(\{\varphi\}, s_1) \wedge ver_{Pr}(\{\varphi\}, s_2)$$
$$sem_{Pr}(\{\varphi\}, \textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) := sem_{Pr}(\{\varphi\}, s_1) \cup sem_{Pr}(\{\varphi\}, s_2)$$

The sequential composition applies s_1 to a state, and then s_2 to the resulting states:

Definition 5.45 ***Semantics of sequential composition***

$$ver_{Pr}(\{\varphi\}, s_1 \ ; s_2) \iff ver_{Pr}(\{\varphi\}, s_1) \wedge ver_{Pr}(sem_{Pr}(\{\varphi\}, s_1), s_2)$$
$$sem_{Pr}(\{\varphi\}, s_1 \ ; s_2) := sem_{Pr}(sem_{Pr}(\{\varphi\}, s_1), s_2)$$

The behaviour of skip is standard:

Definition 5.46 ***Semantics of skip***

$$ver_{Pr}(\{\varphi\}, \textbf{skip}) \iff \top$$
$$sem_{Pr}(\{\varphi\}, \textbf{skip}) := \{\varphi\}$$

5.7.2 Custom interface

Finally, we impose some requirements on the behavior of *custom* statements.

Requirement 5.47 ***General requirements for custom statements***

1. $wfCustom_{Pr}(o) \wedge \varphi' \in semanticsCustom_{Pr}(\varphi, o)$
   $\implies \sigma(\varphi) \subseteq \sigma(\varphi') \wedge \sigma(\varphi') \subseteq \sigma(\varphi) \cup modifCustom(o)$
2. $modifCustom(o) \subseteq readCustom(o)$

The first general requirement states that a *custom* statement cannot undefine variables, and that it modifies the variables it defines. The second one imposes that modifying a variable counts as reading a variable.

We then impose requirements on the consequences of renaming *custom* statements:

Requirement 5.48 *Rename requirements for custom statements*
Assume $wfRenaming(t)$. *Then*

1. $readCustom(renameCustom(t,o)) = \{rename(t,x) | x \in readCustom(o)\}$
2. $modifCustom(renameCustom(t,o)) = \{rename(t,x) | x \in modifCustom(o)\}$
3. $renameCustom(renameInv(t), renameCustom(t,o)) = o$
4. $semanticsCustom_{Pr}(\varphi, o) \neq Error$
   $\implies semanticsCustom_{Pr}(renameState(t,\varphi), renameCustom(t,o)) \neq Error$
5. $\varphi' \in semanticsCustom_{Pr}(renameState(t,\varphi), renameCustom(t,o))$
   $\implies renameState(renameInv(t), \varphi') \in semanticsCustom_{Pr}(\varphi, o)$
6. $wfCustom_{Pr}(o) \implies wfCustom_{Pr}(renameCustom(t,o))$

The first two requirements describe the variables read and modified by a renamed statement. The third one imposes that we can invert the renaming. The fourth and fifth ones describe how to relate the semantics of a renamed statement to the semantics of the original statement. The last requirement says that renaming a well-formed statement results in a well-formed statement.

5.8 Well-definedness and verification of a program

We have presented the semantics and the requirements on the parameters of the framework. These semantics are defined with respect to a program, and are modular. For example, when verifying a method call, we assume the method verifies on its own. This motivates the following definition expressing what it means for a complete program to verify:

Definition 5.49 *Verification of a program*

$$\begin{aligned} & verProg(Pr) \\ \iff & \forall (m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s) \in Pr.ver_{Pr}(h(\overrightarrow{args}) \oplus h(\overrightarrow{rets}) \oplus Inh(P), s \ ; \ \textbf{exhale}\ Q) \end{aligned}$$

A program verifies if all of its methods verify, assuming that all other methods verify, since method calls are approximated by their contracts.

Remark 5.50 *The verification of a method, as written in the previous definition, is equivalent to verifying the method after* $\overrightarrow{args}$ *and* $\overrightarrow{rets}$ *have been defined, beginning with a state which satisfies the precondition:*

$$\begin{aligned} & ver_{Pr}(h(\overrightarrow{args}) \oplus h(\overrightarrow{ret}) \oplus Inh(P), s \ ; \ \textbf{exhale}\ Q) \\ \iff & ver_{Pr}(\{u\}, \textbf{var}\ \overrightarrow{args} \ ; \ \textbf{var}\ \overrightarrow{rets} \ ; \ \textbf{inhale}\ P \ ; \ s \ ; \ \textbf{exhale}\ Q) \end{aligned}$$

However, we can only really speak about the verification of a program when it is well-formed, that is when all of its statements are well-formed. We define these notions as follows:

Definition 5.51 ***Well-formed statements***

$$
\begin{aligned}
& wfStmt_{Pr}(s_1 \,;\, s_2) \iff wfStmt_{Pr}(s_1) \wedge wfStmt_{Pr}(s_2) \\
\mid\ & wfStmt_{Pr}(\textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) \iff wfStmt_{Pr}, s_1) \wedge wfStmt_{Pr}(s_2) \\
\mid\ & wfStmt_{Pr}(\textbf{while } (b) \textbf{ inv } I \ \{s\}) \iff wfAssert(I) \wedge wfStmt_{Pr}(s) \\
\mid\ & wfStmt_{Pr}(\vec{y} := m(\vec{x})) \iff (\exists \overrightarrow{args}, \overrightarrow{rets}, P, Q, s.(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s) \in Pr \\
& \wedge length(\vec{x}) = length(\overrightarrow{args}) \wedge length(\vec{y}) = length(\overrightarrow{rets}) \\
& \wedge \textit{elements of } \vec{x} \textit{ and } \vec{y} \textit{ are distinct}) \\
\mid\ & wfStmt_{Pr}(\textbf{inhale } P) \iff supported(P) \\
\mid\ & wfStmt_{Pr}(\textbf{exhale } P) \iff supported(P) \\
\mid\ & wfStmt_{Pr}(\textbf{custom } o) \iff wfCustom_{Pr}(o) \\
\mid\ & wfStmt_{Pr}(\textbf{skip}) \iff \top
\end{aligned}
$$

A while loop is well-formed if and only if its invariant is well-formed (intuitionistic and supported) and its body is well-formed. No condition is imposed on the loop guard b. A method call is well-formed if the numbers of arguments and return variables provided match the definition of the method. Moreover, for simplicity, we do not allow method calls such as $[x_1, x_2] := m([x_1, x_1, x_2, x_2])$. Indeed, all variables in a method call have to be distinct (this is not the case in Viper, where only return variables have to be distinct). Finally, given our definitions of **inhale** and **exhale** semantics, we want to use only supported assertions. The reason is that the minimal satisfying set of an assertion which is not supported does not have the same meaning. Inhaling general assertions (such as $x.f == 5$) can be encoded via *custom* statements.

We finally define what it means for a program to be well-formed:

Definition 5.52 ***Well-defined methods and programs***
A ***method*** $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s)$ *is* ***well-formed****, written*

$$wfMethod_{Pr}((m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s))$$

if and only if all the following properties are satisfied:

1. *All elements from* $\overrightarrow{args}$ *and* $\overrightarrow{rets}$ *are distinct.*
2. *The precondition is well-formed:* $wfAssert(P)$.
3. *The precondition only deals with arguments:* $readAssert(P) \subseteq \overrightarrow{args}$.

4. *The postcondition is well-formed:* $wfAssert(Q)$
5. *The postcondition only deals with arguments and return variables:* $readAssert(Q) \subseteq \overrightarrow{args} \cup \overrightarrow{rets}$.
6. *The arguments are not modified:* $\overrightarrow{args} \cap modif(s) = \varnothing$.
7. *The method body is well-formed:* $wfStmt_{Pr}(s)$.

A ***well-formed program*** *is defined as follows:*

$$wfProg(Pr) \iff (\forall method \in Pr.\ wfMethod_{Pr}(method))$$

We have now finished the definition of our framework. Given a separation algebra $(\Sigma, \oplus, u, C)$, a type V of variable names, a type O to encode *custom* statements, and four interfaces (store, assertions, rename and custom) which satisfy some requirements, we have defined a language with clearly defined semantics. We can now go back to the core of this thesis, inlining, express a soundness condition, and finally prove soundness of inlining in this framework.

Chapter 6

Soundness of Static Inlining

The previous two chapters define the framework we work with. This chapter deals with the soundness of inlining in this framework, as sketched in Chapter 3. We first formally define the **mono** and **framing** properties (introduced in terms of Viper in Chapter 3). We then define inlining up to a bound and we express the soundness condition (i.e., the condition under which soundness of inlining holds). The last four sections deal with the proof of the soundness. Section 6.3 describes the soundness theorem, as well as the soundness invariant we use to prove this theorem (by induction). We only present the method call case and the loop case of the proof by induction, respectively in Section 6.4 and Section 6.5, since they are the two most interesting proofs. Everything we present in this chapter has been mechanized and proved with the proof assistant Isabelle/HOL [16] (see Appendix A).

6.1 Mono and framing

As sketched in chapter 3, **mono** is the combination of **safeMono** and **monoOut**. These concepts are easy to define in our separation algebra:

Definition 6.1 ***safeMono, monoOut and mono***

$$\begin{aligned} \mathit{safeMono}_{Pr}(s) \iff & (\forall A, B \subseteq \Sigma.(B >> A \land \mathit{ver}_{Pr}(A, s)) \Rightarrow \mathit{ver}_{Pr}(B, s)) \\ \mathit{monoOut}_{Pr}(s) \iff & (\forall A, B \subseteq \Sigma.(B >> A \land \mathit{ver}_{Pr}(A, s) \land \mathit{ver}_{Pr}(B, s)) \\ & \implies \mathit{sem}_{Pr}(B, s) >> \mathit{sem}_{Pr}(A, s)) \\ \mathit{mono}_{Pr}(s) \iff & \mathit{safeMono}_{Pr}(s) \land \mathit{monoOut}_{Pr}(s) \end{aligned}$$

safeMono simply expresses that the function $\lambda A.\mathit{ver}_{Pr}(A, s)$ is non-decreasing,[1] while *monoOut* expresses that the function $\lambda A.\mathit{sem}_{Pr}(A, s)$ is non-decreasing[2]

[1] The function preserves validity for stronger states.

[2] The function returns a stronger (or equally strong) set of output states for a set of stronger input states.

($\lambda A.semPr(A,s)$ is defined on A when $ver_{Pr}(A,s)$).

We defined **framing** as respecting the frame rule. That is, when we have a state which verifies with a framing statement, if we add another state which does not interfere with the statement, then we get at least this other state afterwards. This is expressed formally as follows:

Definition 6.2 *framing*

$$framing_{Pr}(s) \iff mono_{Pr}(s) \wedge (\forall \varphi, r \in \Sigma.ver_{Pr}(\{\varphi\}, s) \wedge modif(s) \cap \sigma(r) = \varnothing \implies sem_{Pr}(\{\varphi \oplus r\}, s) >> sem_{Pr}(\{\varphi\}, s) \oplus \{r\})$$

A **framing** statement is **mono**. The non-interference condition, saying that r does not define any variable which is modified by s, is expressed as $modif(s) \cap \sigma(r) = \varnothing$. Then, for any two states φ and r such that s verifies with φ and r does not interfere with s, executing s from the state $\varphi \oplus r$ results in a set of states stronger than what we would get by adding r to all[3] output states when s is executed from the state φ.

6.2 Formalization of soundness

Now that we have expressed **mono** and **framing**, we can formally express the soundness condition, sketched in Chapter 3. We first define formally what it means to inline up to a bound.

6.2.1 Inlining

We first define inlining for all cases but method calls, which are a bit more complicated. The inline function *inl* is parametrized by a program Pr (to handle method calls), and a depth bound n. It takes as input the statement s to inline, but also a list of variable names $\vec{l}$ which are already defined elsewhere in the inlined program, and have thus to be avoided when inlining method calls, to avoid interferences (see Definition 6.7).

[3]The sum of a set of states and a singleton $\{r\}$ does not exactly results in the set where all states have been added to r. States which cannot be added to r are simply removed.

Definition 6.3 ***Inline (except one method call case)***

$$
\begin{array}{ll}
inl^0_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) & := \textbf{assume } \bot \\
inl^0_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) & := \textbf{assume } \neg b \\
inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) & := \ldots \textit{(see below)} \\
inl^{n+1}_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) & := \textbf{if } (b) \ \{inl^n_{Pr}(\vec{l}, s) \ ; \ inl^n_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\})\} \\
inl^n_{Pr}(\vec{l}, s_1 \ ; s_2) & := inl^n_{Pr}(\vec{l}, s_1) \ ; inl^n_{Pr}(\vec{l}, s_2) \\
inl^n_{Pr}(\vec{l}, \textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) & := \textbf{if } (*) \ \{inl^n_{Pr}(\vec{l}, s_1)\} \textbf{ else } \{inl^n_{Pr}(\vec{l}, s_2)\} \\
inl^n_{Pr}(\vec{l}, s) & := s \qquad \text{(otherwise)}
\end{array}
$$

We begin the inlining with a bound n. Everytime we inline a loop or a method call (see below), we decrease this bound by 1 and recursively call the inline function. When we have already inlined up to a depth of n, we do not go further. For method calls, we simply stop the execution by assuming $\bot$. In the case of loops, we simply assume $\neg b$, which is equivalent to assuming $b \Rightarrow \bot$. Indeed, if b is true, then we would have had to go on with the loop, but otherwise we are out of the loop.

To inline a loop (when we have not reached the depth bound yet), we use the deterministic **if** construct (which, as explained in Definition 5.3, can be encoded using the non-deterministic $\textbf{if}(*)$). If b is true, then we enter the loop and we first inline one iteration of the loop (i.e., the loop body). After this inlined iteration, we are still in the loop, so we inline the loop (but with the depth decreased by one). If b is false, then we do nothing. Finally, inlining sequential compositions and conditional branchings is straightforward.

Remark 6.4 *The definition of $inl^n_{Pr}(\vec{l}, s_1 \ ; s_2)$ uses the same list of variable names $\vec{l}$ for the two recursive calls (inlining of s_1 and s_2), instead of recording the newly defined variables in $inl^n_{Pr}(\vec{l}, s_1)$. Variable names can therefore collide and variables can be captured. This will be modified in the future.*

We now need to define the inlining of a method call when the depth bound has not been reached. This is also related to the list of variables used by the inlining function. To define this inlining, we first need to define what we mean by "variables read by a statement":

Definition 6.5 *Variables read by a statement*

$$
\begin{aligned}
& read(s_1\ ;\ s_2) = read(s_1) \cup read(s_2) \\
\mid\ & read(\textbf{if}\ (*)\ \{s_1\}\ \textbf{else}\ \{s_2\}) = read(s_1) \cup read(s_2) \\
\mid\ & read(\vec{y} := m(\vec{x})) = \vec{x} \cup \vec{y} \\
\mid\ & read(\textbf{while}\ (b)\ \textbf{inv}\ I\ \{s\}) = read(s) \cup readAssert(b) \cup readAssert(I) \\
\mid\ & read(\textbf{assume}\ P) = readAssert(P) \\
\mid\ & read(\textbf{inhale}\ P) = readAssert(P) \\
\mid\ & read(\textbf{exhale}\ P) = readAssert(P) \\
\mid\ & read(\textbf{var}\ \vec{l}) = \vec{l} \\
\mid\ & read(\textbf{havoc}\ \vec{l}) = \vec{l} \\
\mid\ & read(\textbf{skip}) = \varnothing \\
\mid\ & read(\textbf{custom}\ o) = readCustom(o)
\end{aligned}
$$

This definition is similar to the definition of the $modif$ function (Definition 5.32), with some differences. Since we care about all variables which are read, we include all variables read by assertions, as well as variables passed as arguments when calling a method. Since we imposed as a requirement that $readCustom(o) \subseteq modifCustom(o)$, we can show the following lemma:

Lemma 6.6 *Every variable that is modified is also read*

$$modif(s) \subseteq read(s)$$

Using this *read* function, we can now express how we inline method calls (we use $++$ to denote the concatenation of two lists):

Definition 6.7 *Inlining of method calls*

***First case**: The method is defined, i.e., there exists* $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s) \in Pr$. *Let* $s' = rename\left((\overrightarrow{args} ++ \overrightarrow{rets}, \vec{x} ++ \vec{y}, \vec{l}, read(s)), s\right)$, *and* $\vec{l'} = \vec{l} \cup read(s')$. *Then*

$$inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) := inl^{n}_{Pr}(\vec{l'}, s')$$

***Second case**: The method is undefined.*

$$inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) := \textbf{exhale}\ \bot$$

In the case of a method call not well-formed (if the method called is not defined), we simply exhale $\bot$, which fails with any state. It does not really matter, since the soundness we prove (see Section 6.3) assumes well-formedness, which in particular means that every method call is well-formed, that is the method which is called is defined. When the method call is well-formed, we

Listing 6.1: Loop condition not **mono**.

```
1 var b: Bool := false
2 while (perm(x.f) <= 1/2) {
3     b := true
4     ...
5 }
6 assert b
```

Listing 6.2: Negated loop condition not **mono**.

```
1 var b: Bool := true
2 while (perm(x.f) >= 1/2) {
3     b := false
4     ...
5 }
6 assert b
```

Figure 6.1: Loops with conditions not well-formed monotonic

take the body of the method, rename it by modifying the arguments and the return variables using the ones provided in the method calls.

The meaning of the renaming quadruple $(\overrightarrow{args}{+}{+}\overrightarrow{rets}, \vec{x}{+}{+}\vec{y}, \vec{l}, read(s))$ is the following:

1. $\overrightarrow{args}$ variables are renamed to $\vec{x}$ variables.
2. $\overrightarrow{rets}$ variables are renamed to $\vec{y}$ variables.
3. $\vec{l}$ variables are avoided: If a variable has a name in $\vec{l}$, then it is renamed to a variable with a name not in $\vec{l}$.
4. Only variables from $read(s)$ will be renamed by the functions using this quadruple. Therefore, the property to avoid variables is satisfied for these variables.

We also need to record the new variables defined by the renamed method body, to not capture them while inlining deeper. This is the meaning of $\vec{l'} = \vec{l} \cup read(s')$: We add the variables read by s' to $\vec{l'}$. This new list $\vec{l'}$ is then used in the recursive call to the inline function, for inlining the renamed method body.

6.2.2 Soundness condition

The sketch of the soundness described in Chapter 3 said the following:

1. Method and loop bodies have to be **framing**.
2. Non-inlinable statements, that is statements which do not contain method calls and loops, have to be **mono**.

We need to formally clarify several things. First, we only looked at examples of method calls. In the case of loops, we also have a loop condition b. This condition cannot be any assertion.

Indeed, we show in Figure 6.1 two examples[4] where all statements (without considering loop conditions) are **framing**, but we do not have soundness. Listing 6.1 is not sound for inlining, since a state with no permission to $x.f$ verifies, but a state with full permission does not. In this case, `assume perm(x.f) <= 1/2` is not **mono**. Listing 6.2 shows another example where inlining is unsound. In this other case, `assume !(perm(x.f >= 1/2))` is not **mono**. Formally, the loop condition has to satisfy the following property:

Definition 6.8 ***Well-formed monotonic assertions***

$$wfm_{Pr}(b) \iff mono_{Pr}(\textbf{assume } b) \land mono_{Pr}(\textbf{assume } \neg b)$$

It should be clear from the examples why we require this. What this means is that, when an assertion is well-defined in a state, then it will have the same value (true or false) for any stronger state. Another reason is that, when we inline a loop iteration, we use a deterministic if, which is translated to a non-deterministic if using **assume** b and **assume** $\neg b$. $mono_{Pr}(\textbf{assume } b)$ (and especially **monoOut**) ensures that a stronger state will enter the loop if a weaker state enters the loop, whereas $mono_{Pr}(\textbf{assume } \neg b)$ (also especially **monoOut**) ensures that a stronger state will avoid the loop if a weaker state avoids the loop.

Moreover, we informally speak about non-inlinable statements. What we mean is statements which do not contain any loop nor any method call. Formally:

Definition 6.9 ***Inlinable***

$$\begin{aligned}
&inlinable(\vec{y} := m(\vec{x})) && \iff \top \\
&inlinable(\textbf{while } (b) \textbf{ inv } I \ \{s\}) && \iff \top \\
&inlinable(s_1 \ ; s_2) && \iff inlinable(s_1) \lor inlinable(s_2) \\
&inlinable(\textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) && \iff inlinable(s_1) \lor inlinable(s_2) \\
&inlinable(s) && \iff \bot \qquad \text{(otherwise)}
\end{aligned}$$

One simple consequence of this definition is the following:

Lemma 6.10 $\neg inlinable(s) \implies inl^n_{Pr}(\vec{l}, s) = s$

A statement not inlinable stays the same when inlined. We now have all the elements to formally express the soundness condition.

The soundness condition takes a statement s, the list of read variables $\vec{l}$ it is inlined with, the program Pr and also the remaining depth n. Recall that the

[4] These are not possible Viper programs, since Viper forbids the use of `perm` in loop conditions.

list $\vec{l}$ records the variable names which are already defined somewhere in the inlined program, and thus the variable names which should be avoided when renaming a method body, to avoid capturing variables.

Definition 6.11 ***Soundness Condition***
If $\neg inlinable(s)$ ***then*** $SC^n_{Pr}(\vec{l}, s) \iff mono_{Pr}(s)$.

Otherwise for inlinable statements:

$$
\begin{array}{lll}
SC^0_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) & \iff mono_{Pr}(\textbf{assume } \neg b) & (*) \\
SC^n_{Pr}(\vec{l}, s_1 \ ; s_2) & \iff SC^n_{Pr}(\vec{l}, s_1) \wedge SC^n_{Pr}(\vec{l}, s_2) & \\
SC^n_{Pr}(\vec{l}, \textbf{if } (*) \ \{s_1\} \textbf{ else } \{s_2\}) & \iff SC^n_{Pr}(\vec{l}, s_1) \wedge SC^n_{Pr}(\vec{l}, s_2) & \\
SC^n_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) & \iff \ldots \textit{(see below)} & \\
SC^{n+1}_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) & \iff (framing_{Pr}(inl^n_{Pr}(\vec{l}, s)) \vee framing_{Pr}(s)) \wedge wfm_{Pr}(b) & \\
 & \quad \wedge SC^n_{Pr}(\vec{l}, s) \wedge SC^n_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\}) & \\
SC^0_{Pr}(\vec{l}, s) & \iff \top & \text{(otherwise)}
\end{array}
$$

Remark 6.12 ***(*)****: In the Isabelle formalization (Appendix A), we require*

$$SC^0_{Pr}(\textbf{while } (b) \textbf{ inv } I \ \{s\}) \iff wfm_{Pr}(b) \wedge mono_{Pr}(s)$$

because we use in the Isabelle proof the lemma

$$SC^n_{Pr}(s) \implies mono_{Pr}(s)$$

This lemma does not hold without the additional requirement (one reason being that the body inside the loop can be not ***safeMono****, which makes the loop not* ***safeMono****). However, this lemma is not necessary for the proof. Indeed, we do not use this lemma in the proofs presented here, and the Isabelle formalization should be modified in this direction in the future.*

We define the missing method call case below (for $n = 0$, there is no condition to be satisfied). Let us explain the intuition behind the soundness condition for loops, given in Definition 6.11. Let us first consider a loop with condition b. If $n = 0$, then the soundness condition requires that **assume** $\neg(b)$ must be **mono** (which should be clear from our discussion above). However, when the remaining depth is not zero but $n + 1$, we require several properties:

1. $wfm_{Pr}(b)$: As explained above, we use both **assume** b and **assume** $\neg b$.
2. $SC^n_{Pr}(\vec{l}, s)$: We need this property since we inline the loop body.
3. $SC^n_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\})$: We need this property since we also inline the loop (with a smaller bound).

4. $framing_{Pr}(inl^n_{Pr}(\vec{l}, s)) \vee framing_{Pr}(s)$: Interestingly, this means that it is sufficient that either the original loop body is **framing** (with any existing annotation) or the inlined loop body is **framing**.

We finally define the soundness condition for method calls:

Definition 6.13 ***Soundness condition for method calls***
First, the soundness condition for a method call implies that there exist $\overrightarrow{args}$, $\overrightarrow{rets}$, P, Q *and method body* s *such that*

$$(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s) \in Pr$$

In this case, let $s' = rename\left((\overrightarrow{args}{+}{+}\overrightarrow{rets}, \vec{x}{+}{+}\vec{y}, \vec{l}, read(s)), s\right)$, *and* $\vec{l'} = \vec{l} \cup read(s')$. *Then*

$$SC^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) \iff (framing_{Pr}(inl^n_{Pr}(\vec{l'}, s')) \vee framing_{Pr}(s)) \wedge SC^n_{Pr}(\vec{l'}, s')$$

Since we inline by renaming the method body and recursively calling the inline function on s' (the renamed method body) and $\vec{l'}$ (the new list of variables to avoid), we express the soundness condition using them. As in the case of loop, we have the disjunction $framing_{Pr}(inl^n_{Pr}(\vec{l'}, s')) \vee framing_{Pr}(s)$. We also need $SC^n_{Pr}(\vec{l'}, s')$ since this is what we inline.

One important and useful lemma which can be proved by computational induction on the structure of the soundness condition is the following:

Lemma 6.14

$$SC^n_{Pr}(\vec{l}, s) \implies mono_{Pr}(inl^n_{Pr}(\vec{l}, s))$$

Remark 6.15 *It is interesting to note that we have neither* $framing_{Pr}(inl^n_{Pr}(\vec{l}, s)) \implies framing_{Pr}(s)$, *nor the reciprocal. It is easy to see, for example, that we can have* $framing_{Pr}(inl^0_{Pr}(\vec{l}, s))$ *without having* $framing_{Pr}(s)$. *On the other hand, take the Viper method call* `s := y := m(x)`, *with the body of m being* `exhale acc(x.f, perm(x.f))`. *We know that the body of m is not framing, therefore we have* $\neg framing_{Pr}(inl^1_{Pr}(\vec{l}, s))$. *However, any well-formed method call is framing (because of the semantics of method calls).*

6.3 Soundness theorem

We now have all the formalism needed to express the soundness theorem:

Theorem 6.16 ***Soundness***
Assume

1. $wfProg(Pr)$

2. $wfStmt_{Pr}(s)$
3. $SC^n_{Pr}(modif(s), s)$
4. $verProg(Pr)$
5. $ver_{Pr}(u, s)$ *(u is the empty state of the separation algebra).*

Then $ver_{Pr}(u, inl^n_{Pr}(modif(s), s))$.

Remark 6.17 ***Contraposition of the soundness theorem***
By contraposition, we obtain that, under the following conditions:

1. $wfProg(Pr)$
2. $wfStmt_{Pr}(s)$
3. $SC^n_{Pr}(modif(s), s)$

If

$$\exists n. \neg ver_{Pr}(u, inl^n_{Pr}(modif(s), s))$$

then there does not exist any way of annotating the program such that both $verProg(Pr)$ *and* $ver_{Pr}(u, s)$.

The contraposition corresponds to what was discussed in Chapter 1. Namely, if the inlined program does not verify, then there exists a *fundamental error* in the original program.

We do not prove the soundness theorem directly, but we define and prove a stronger property we call the soundness invariant. We first define the domain of a set of states, needed to express the invariant.

Definition 6.18 ***Domain of a set of states***

$$domain(A) := \cup_{\varphi \in A} \sigma(\varphi)$$

The domain of a set is simply the set of variables defined by all of its states. We use this definition to express the fact that, when inlining method calls, we rename the variables of the corresponding method bodies to avoid capturing variables already defined.

Definition 6.19 ***Partial Soundness Property***

$$\begin{aligned} PSP^n_{Pr}(\vec{l}, s) := (\forall A', A \subseteq \Sigma. A' >> A \wedge domain(A') \subseteq \vec{l} \wedge ver_{Pr}(A, s) \\ \Longrightarrow ver_{Pr}(A', inl^n_{Pr}(\vec{l}, s)) \wedge sem_{Pr}(A', inl^n_{Pr}(\vec{l}, s)) >> sem_{Pr}(A, s)) \end{aligned}$$

This partial soundness property is parametrized by a program Pr, a bound n, the list of variables $\vec{l}$ which records variables already defined somewhere in the program, and a statement s. $domain(A') \subseteq \vec{l}$ asserts that all variables

defined by all states of A are included in the list of variables $\vec{l}$. This is to avoid interferences with newly defined variables when inlining the program. In the end, we instantiate this partial soundness property for $A' = A := \{u\}$, a singleton with the empty state, for which this property is trivially satisfied for any list $\vec{l}$. The partial soundness property says that, for any two sets of states A' and A such that $A' >> A$, if A' does not interfere with $\vec{l}$ as we just described, and if s verifies with the set of states A, then A' verifies with the inlined statement. Moreover, executing the inlined statement from the set of states A' results in a stronger set of states than executing the original statement from the set of states A.

This is intuitively what happens under the soundness condition: Loop invariants and method preconditions and postconditions allow hiding some information (such as the values of variables) and leak some permissions. On the other hand, all the information and the permissions are visible in the inlined program.

We now define the soundness invariant we prove, using this partial soundness property:

Definition 6.20 ***Soundness Invariant*** *(induction hypothesis)*

$$SI^n_{Pr}(\vec{l}, s) \iff \Big(wfStmt_{Pr}(s) \wedge modif(s) \subseteq \vec{l} \wedge SC^n_{Pr}(\vec{l}, s) \implies PSP^n_{Pr}(\vec{l}, s)\Big)$$

That is, we prove that for all well-formed statements such that the variables it modifies are included in $\vec{l}$, under the soundness condition, we have the partial soundness property. Formally, we simply prove

Lemma 6.21

$$SI^n_{Pr}(\vec{l}, s)$$

That is, this soundness invariant is true for any program Pr, any natural number n, any list of variable names $\vec{l}$ and any statement s. We prove this lemma by computational induction on the structure of the inline function. The complete proof has been formalized in Isabelle/HOL. However, we only show in this chapter two interesting cases of the proof, namely loops and method calls. By instantiating this soundness invariant for $\vec{l} := modif(s)$, we get Theorem 6.16, the soundness theorem.

6.4 Induction case: method calls

In this section, we prove the following induction case, for method calls:

Lemma 6.22 ***Method calls induction case***
Assume

1. $wfProg(Pr) \wedge verProg(Pr)$
2. $(m, \overrightarrow{args}, \overrightarrow{rets}, P, Q, s) \in Pr$
3. $s' = rename((\overrightarrow{args}{+}{+}\overrightarrow{rets}, \vec{x}{+}{+}\vec{y}, \vec{l}, read(s)), s)$
4. $\vec{l'} = \vec{l} \cup read(s')$
5. $SI^n_{Pr}(\vec{l'}, s')$

Then $SI^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$.

Proof To prove $SI^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$, we assume $wfStmt_{Pr}(\vec{y} := m(\vec{x}))$, $modif(\vec{y} := m(\vec{x})) \subseteq \vec{l}$, and $SC^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$, and we show $PSP^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$.

We get, by definitions, $wfStmt_{Pr}(s)$ and $\vec{y} \subseteq \vec{l}$. Using $verProg(Pr)$ and that the method call is well-formed, we also get

$$ver_{Pr}(\{u\}, \textbf{var } (\overrightarrow{args}{+}{+}\overrightarrow{rets}) \textbf{ ; inhale } P \textbf{ ; } s \textbf{ ; exhale } Q)$$

Let $t := (\overrightarrow{args}{+}{+}\overrightarrow{rets}, \vec{x}{+}{+}\vec{y}, \vec{l}, read(s))$, which is well-formed. Then let $P' := rename(P, t)$ and $Q' := rename(Q, t)$ be the renaming of the precondition and postcondition with respect to the renaming quadruple t.

From $SC^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$ we get $SC^n_{Pr}(\vec{l'}, s')$. Then, from $SI^n_{Pr}(\vec{l'}, s')$ and the properties of well-formed renaming we get $PSP^n_{Pr}(\vec{l'}, s')$ and

$$ver_{Pr}(\{u\}, \textbf{var } (\vec{x}{+}{+}\vec{y}) \textbf{ ; inhale } P' \textbf{ ; } s' \textbf{ ; exhale } Q')$$

which gives

$$ver_{Pr}(h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P'), s')$$

and

$$sem_{Pr}(h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P'), s') >> Inh(Q')$$

Towards proving $PSP^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$, let $A', A \subseteq \Sigma$ such that $A' >> A$, $domain(A') \subseteq \vec{l}$ and $ver_{Pr}(A, \vec{y} := m(\vec{x}))$. Let then $\varphi' \in A'$, $\varphi \in A$ such that $\varphi << \varphi'$. We have

$$ver_{Pr}(\{\varphi\}, \textbf{exhale } P' \textbf{ ; havoc } \vec{y} \textbf{ ; inhale } Q')$$

We then can obtain $i \in Inh(P')$, $r \in \Sigma$ such that $\varphi = i \oplus r$. Let $r' := \overline{h}(r, \vec{y})$, we have then $\varphi = i \oplus |\varphi| \oplus r'$.

Let us now describe the semantics of $\vec{y} := m(\vec{x})$:

$$\begin{aligned}
sem_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x})) &= sem_{Pr}(\{i \oplus |\varphi| \oplus r'\}, \vec{y} := m(\vec{x})) \\
&= sem_{Pr}(\{i \oplus |\varphi| \oplus r'\}, \textbf{exhale } P' \textbf{ ; havoc } \vec{y} \textbf{ ; inhale } Q') \\
&= sem_{Pr}(\{|i| \oplus |\varphi| \oplus r'\}, \textbf{havoc } \vec{y} \textbf{ ; inhale } Q') \\
&= sem_{Pr}(\{\overline{h}(|i| \oplus |\varphi| \oplus r', \vec{y})\} \oplus h(\vec{y}), \textbf{inhale } Q') \\
&= sem_{Pr}(\{\overline{h}(|\varphi|, \vec{y}) \oplus r'\} \oplus h(\vec{y}), \textbf{inhale } Q') \\
&= \{\overline{h}(|\varphi|, \vec{y})\} \oplus \{r'\} \oplus h(\vec{y}) \oplus Inh(Q')
\end{aligned}$$

We know $inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})) = inl^{n}_{Pr}(\vec{l'}, s')$. Our aim is to prove two things:

1. $H_1 : ver_{Pr}(\{\varphi\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})))$
2. $H_2 : sem_{Pr}(\{\varphi\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))) >> sem_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x}))$

These two results are enough to conclude the proof. Indeed, we get

$$mono_{Pr}(inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})))$$

from Lemma 6.14. Using this property, combined with $\varphi << \varphi'$, we get

1. $ver_{Pr}(\{\varphi'\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})))$
2. $sem_{Pr}(\{\varphi'\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))) >> sem_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x}))$

which concludes the proof.

We prove therefore these two properties, H_1 and H_2, by considering the two cases corresponding to the disjunction of the soundness condition

$$SC^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))$$

Case 1: $framing_{Pr}(s)$.

In this case, we also have $framing_{Pr}(s')$ since the renaming quadruple t is well-formed and s' is the renamed version of s.

Let $B' := \{\varphi\}$ and $B := h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P') \oplus \{r'\}$. We have $B' >> B$. Indeed, $\varphi = i \oplus |\varphi| \oplus r'$, with $i \in Inh(P')$, and $\vec{x} \cup \vec{y} \subseteq \sigma(\varphi)$. Moreover, we have $ver_{Pr}(B, s')$, using $mono_{Pr}(s')$ (which holds since s' is **framing**) and $ver_{Pr}(h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P'), s')$. We also have $domain(B') = \sigma(\varphi) \subseteq \vec{l} \subseteq \vec{l'}$.

By applying $PSP^{n}_{Pr}(\vec{l'}, s')$ we get

$$ver_{Pr}(B', inl^{n}_{Pr}(\vec{l'}, s'))$$

and

$$sem_{Pr}(B', inl^n_{Pr}(\vec{l'}, s')) >> sem_{Pr}(B, s')$$

Then

$$\begin{aligned}
&sem_{Pr}(\{\varphi\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})))\\
&= sem_{Pr}(B', inl^n_{Pr}(\vec{l'}, s'))\\
&>> sem_{Pr}(B', inl^n_{Pr}(\vec{l'}, s')) \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus h(\vec{y})\\
&\qquad\qquad \text{(since } \vec{y} \subseteq \sigma(\varphi) \text{ and } modif(s') \cap \sigma(\varphi) \subseteq \vec{y}\text{)}\\
&>> sem_{Pr}(B, s') \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus h(\vec{y})\\
&= sem_{Pr}(h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P') \oplus \{r'\}, s') \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus h(\vec{y})\\
&>> sem_{Pr}(h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P'), s') \oplus \{r'\} \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus h(\vec{y})\\
&\qquad\qquad \text{(since } s' \text{ is } \textbf{framing}\text{)}\\
&>> Inh(Q') \oplus \{r'\} \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus h(\vec{y})\\
&= sem_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x}))
\end{aligned}$$

Case 2: $framing_{Pr}(inl^n_{Pr}(\vec{l'}, s'))$.

Let $B' := \{i \oplus |\varphi|\}$ and $B := h(\vec{x}) \oplus h(\vec{y}) \oplus Inh(P')$. We have $B' >> B$, since $i \in Inh(P')$. We also have $domain(B') \subseteq \sigma(\varphi) \subseteq \vec{l} \subseteq \vec{l'}$ and $ver_{Pr}(B, s')$. By applying $PSP^n_{Pr}(\vec{l'}, s')$ we get

$$ver_{Pr}(B', inl^n_{Pr}(\vec{l'}, s'))$$

and

$$sem_{Pr}(B', inl^n_{Pr}(\vec{l'}, s')) >> sem_{Pr}(B, s') >> Inh(Q')$$

We have $mono_{Pr}(inl^n_{Pr}(\vec{l'}, s'))$ since it is **framing**. Therefore, since $\{\varphi\} >> B'$, we have $ver_{Pr}(\{\varphi\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x})))$.

Moreover

$$
\begin{aligned}
& sem_{Pr}(\{\varphi\}, inl^{n+1}_{Pr}(\vec{l}, \vec{y} := m(\vec{x}))) \\
& = sem_{Pr}(B' \oplus \{r'\}, inl^{n}_{Pr}(\vec{l'}, s')) \\
& = sem_{Pr}(B' \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus \{r'\}, inl^{n}_{Pr}(\vec{l'}, s')) \\
& >> sem_{Pr}(B', inl^{n}_{Pr}(\vec{l'}, s')) \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus \{r'\} \\
& \qquad\qquad \text{(since } inl^{n}_{Pr}(\vec{l'}, s') \text{ is } \textbf{framing}\text{)} \\
& >> sem_{Pr}(B, s') \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus \{r'\} \\
& >> sem_{Pr}(B, s') \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus \{r'\} \oplus h(\vec{y}) \\
& >> Inh(Q') \oplus \{\overline{h}(|\varphi|, \vec{y})\} \oplus \{r'\} \oplus h(\vec{y}) \\
& = sem_{Pr}(\{\varphi\}, \vec{y} := m(\vec{x})) \qquad \square
\end{aligned}
$$

6.5 Induction case: loops

We prove, in this section, the following induction case, for loops:

Lemma 6.23 ***Loops induction case***
Assume

1. $wfProg(Pr) \wedge verProg(Pr)$
2. $SI^{n}_{Pr}(\vec{l}, s)$
3. $SI^{n}_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\})$

Then $SI^{n+1}_{Pr}(\vec{l}, \textbf{while } (b) \textbf{ inv } I \ \{s\})$.

Proof Let $w := \textbf{while } (b) \textbf{ inv } I \ \{s\}$.

To prove $SI^{n+1}_{Pr}(\vec{l}, w)$, we assume $wfStmt_{Pr}(w)$, $modif(w) \subseteq \vec{l}$, and $SC^{n+1}_{Pr}(\vec{l}, w)$, and we show $PSP^{n+1}_{Pr}(\vec{l}, w)$.

We then get, by definitions, $wfStmt_{Pr}(s)$, $modif(s) \subseteq \vec{l}$, $SC^{n}_{Pr}(\vec{l}, w)$, and $SC^{n}_{Pr}(\vec{l}, s)$. From 2 and 3 we then obtain $PSP^{n}_{Pr}(\vec{l}, s)$ and $PSP^{n}_{Pr}(\vec{l}, w)$.

Assume $wfStmt_{Pr}(w)$, $modif(w) \subseteq \vec{l}$, and $SC^{n+1}_{Pr}(\vec{l}, w)$. We then get, by definitions, $wfStmt_{Pr}(s)$, $modif(s) \subseteq \vec{l}$, $SC^{n}_{Pr}(\vec{l}, w)$, and $SC^{n}_{Pr}(\vec{l}, s)$. From 2 and 3 we then obtain $PSP^{n}_{Pr}(\vec{l}, s)$ and $PSP^{n}_{Pr}(\vec{l}, w)$.

Towards proving $PSP^{n+1}_{Pr}(\vec{l}, w)$, let $A', A \subseteq \Sigma$ such that $A' >> A$, $domain(A') \subseteq \vec{l}$, and $ver_{Pr}(A, w)$. Let then $\varphi' \in A'$, $\varphi \in A$ such that $\varphi << \varphi'$. We then have $\sigma(\varphi) \subseteq \vec{l}$, $\sigma(\varphi') \subseteq \vec{l}$, and $ver_{Pr}(\{\varphi\}, w)$.

Our aim is to prove two things:

1. $ver_{Pr}(\{\varphi'\}, inl_{Pr}^{n+1}(\vec{l}, w))$
2. $sem_{Pr}(\{\varphi'\}, inl_{Pr}^{n+1}(\vec{l}, w)) >> sem_{Pr}(\{\varphi\}, w)$

Moreover,

$$\begin{aligned} inl_{Pr}^{n+1}(\vec{l}, w) &= \textbf{if } (b)\ \{inl_{Pr}^{n}(\vec{l}, s)\ ;\ inl_{Pr}^{n}(\vec{l}, w)\} \\ &= \textbf{if } (*)\ \{\textbf{assume } b\ ;\ inl_{Pr}^{n}(\vec{l}, s)\ ;\ inl_{Pr}^{n}(\vec{l}, w)\}\ \textbf{else}\ \{\textbf{assume } \neg b\} \end{aligned}$$

Since we have $wfm(b)$, we know that $mono_{Pr}(\textbf{assume } b)$ and $mono_{Pr}(\textbf{assume } \neg b)$. Using the verification of the loop (i.e., $ver_{Pr}(\{\varphi\}, w)$), we get that b and $\neg b$ are well-defined with φ. Therefore, $ver_{Pr}(\{\varphi'\}, \textbf{assume } b)$, and $ver_{Pr}(\{\varphi'\}, \textbf{assume } \neg b)$. We also get that $\varphi >> Inh(I)$, thus there is an $i \in Inh(I)$ and an $r \in \Sigma$ such that $\varphi = i \oplus r$.

Let $V := modif(s) \cap \sigma(\varphi)$. Let $r' = \overline{h}(r, \vec{V})$, then we have $\varphi = i \oplus |\varphi| \oplus r'$. Let $F := Inh(I) \oplus h(\vec{V}) \oplus \{\overline{h}(|\varphi|, \vec{V})\} \oplus \{r'\}$.

Since $V \subseteq \sigma(\varphi)$, we have

$$\{\varphi\} >> F$$

Let us define a function *filter* to filter a set of states with respect to an assertion, such that

$$filter_p(A) := \{a \in A | P(a)\}$$

Using this function, let us now describe the semantics of the loop:

$$\begin{aligned} sem_{Pr}(\{\varphi\}, w) &= sem_{Pr}(\{i \oplus |\varphi| \oplus r'\}, w) \\ &= sem_{Pr}(\{i \oplus |\varphi| \oplus r'\}, \textbf{exhale } I\ ;\ \textbf{havoc } \vec{V}\ ;\ \textbf{inhale } I\ ;\ \textbf{assume } \neg b) \\ &= sem_{Pr}(\{|i| \oplus |\varphi| \oplus r'\}, \textbf{havoc } \vec{V}\ ;\ \textbf{inhale } I\ ;\ \textbf{assume } \neg b) \\ &= sem_{Pr}(\{\overline{h}(|i| \oplus |\varphi| \oplus r', \vec{V})\} \oplus h(\vec{V}), \textbf{inhale } I\ ;\ \textbf{assume } \neg b) \\ &= sem_{Pr}(\{\overline{h}(|\varphi|, \vec{V})\} \oplus \{r'\} \oplus h(\vec{V}), \textbf{inhale } I\ ;\ \textbf{assume } \neg b) \\ &= sem_{Pr}(\{\overline{h}(|\varphi|, \vec{V})\} \oplus \{r'\} \oplus h(\vec{V}) \oplus Inh(I), \textbf{assume } \neg b) \\ &= sem_{Pr}(F, \textbf{assume } \neg b) \\ &= filter_{\neg b}(F) \end{aligned}$$

We now consider two cases, corresponding to the branch φ takes:

Case 1: $\neg b(\varphi)$.

Then $\neg b(\varphi')$ since $wfm(b)$. We have $sem_{Pr}(\{\varphi'\}, \textbf{assume } b) = \varnothing$, which means that the conditional branch with the loop iteration verifies. Since we also have $ver_{Pr}(\{\varphi'\}, \textbf{assume } \neg b)$, we get $ver_{Pr}(\{\varphi'\}, inl_{Pr}^{n+1}(\vec{l}, w))$.

Moreover

$$\begin{aligned} sem_{Pr}(\{\varphi'\}, inl_{Pr}^{n+1}(\vec{l}, w)) &= sem_{Pr}(\{\varphi'\}, \textbf{assume } \neg b) \\ &= \{\varphi'\} \\ &>> \{\varphi\} \\ &>> F \\ &>> filter_{\neg b}(F) \\ &= sem_{Pr}(\{\varphi\}, w) \end{aligned}$$

therefore concluding this case.

Case 2: $b(\varphi)$.

Then $b(\varphi')$ since $wfm(b)$, thus we have

$$ver_{Pr}(\{\varphi'\}, inl_{Pr}^{n+1}(\vec{l}, w)) = ver_{Pr}(\{\varphi'\}, inl_{Pr}^{n}(\vec{l}, s)\ ;\ inl_{Pr}^{n}(\vec{l}, w))$$

and $sem_{Pr}(\{\varphi'\})(inl_{Pr}^{n+1}(\vec{l}, w)) = sem_{Pr}(\{\varphi'\})(inl_{Pr}^{n}(\vec{l}, s)\ ;\ inl_{Pr}^{n}(\vec{l}, w))$.

We show four preliminary results before concluding:

1. $sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) >> \{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)$ and $ver_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s)$

   We know from the verification of the loop that

   $$ver_{Pr}(\{|\varphi|\}, \textbf{havoc } \vec{V}\ ;\ \textbf{inhale } I\ ;\ \textbf{assume } b\ ;\ s\ ;\ \textbf{exhale } I)$$

   We have

   $$\{\varphi\} >> \{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)$$

   and $b(\varphi)$, thus

   $$\begin{aligned} &sem_{Pr}(\{|\varphi|\}, \textbf{havoc } \vec{V}\ ;\ \textbf{inhale } I\ ;\ \textbf{assume } b) \\ =&filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)) \end{aligned}$$

   We then get

   $$ver_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s)$$

   and

   $$sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) >> Inh(I)$$

   Since s cannot undefine variables, and since $modif(s) \subseteq \vec{V}$, we get

   $$\begin{aligned} &sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) \\ >>&\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I) \end{aligned}$$

2. $sem_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s)) >> sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) \oplus \{r'\}$ and $ver_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s))$:

   We know that

$$\begin{aligned}\{i \oplus |\varphi|\} >> & \{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I) \\ >> & filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I))\end{aligned}$$

   Let us consider the two cases corresponding to the disjunction in the soundness condition:

   Case 2.1: $framing_{Pr}(s)$

   By applying $PSP^n_{Pr}(\vec{l}, s)$ to $A' := \{\varphi'\}$ and $A := \{\varphi\}$, we get $ver_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s))$ and

$$\begin{aligned}sem_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s)) >> & sem_{Pr}(\{\varphi\}, s) \\ = & sem_{Pr}(\{i \oplus |\varphi| \oplus r'\}, s) \\ >> & sem_{Pr}(\{i \oplus |\varphi|\}, s) \oplus r' \\ & \qquad \text{(since } s \text{ is } \textbf{framing}\text{)} \\ >> & sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) \oplus \{r'\} \\ & \qquad \text{(since } s \text{ is } \textbf{mono}\text{)}\end{aligned}$$

   Case 2.2: $framing_{Pr}(inl^n_{Pr}(\vec{l}, s))$
   We have $ver_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s)$. By applying $PSP^n_{Pr}(\vec{l}, s)$ to $A' = A := filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I))$, we get

$$\begin{aligned}& sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), inl^n_{Pr}(\vec{l}, s)) \\ >> & sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s)\end{aligned}$$

   and

$$ver_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), inl^n_{Pr}(\vec{l}, s))$$

   thus $ver_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s))$ (since $mono_{Pr}(inl^n_{Pr}(\vec{l}, s))$).

Eventually,

$$
\begin{aligned}
& sem_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s)) \\
>> & sem_{Pr}(\{\varphi\}, inl^n_{Pr}(\vec{l}, s)) && \text{(since } inl^n_{Pr}(\vec{l}, s) \text{ is } \textbf{mono}\text{)} \\
= & sem_{Pr}(\{i \oplus |\varphi| \oplus r'\}, inl^n_{Pr}(\vec{l}, s)) \\
>> & sem_{Pr}(\{i \oplus |\varphi|\}, inl^n_{Pr}(\vec{l}, s)) \oplus \{r'\} && \text{(since } inl^n_{Pr}(\vec{l}, s) \text{ is } \textbf{framing}\text{)} \\
>> & sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), inl^n_{Pr}(\vec{l}, s)) \oplus \{r'\} \\
& && \text{(since } inl^n_{Pr}(\vec{l}, s) \text{ is } \textbf{mono}\text{)} \\
>> & sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) \oplus \{r'\}
\end{aligned}
$$

3. $sem_{Pr}(F, w) >> filter_{\neg b}(F)$ and $ver_{Pr}(F, w)$:
   Let $f \in F$. There is some $i_f \in Inh(I)$ and $h_V \in h(\vec{V})$ such that $f = i_f \oplus h_V \oplus \overline{h}(|\varphi|, \vec{V}) \oplus r'$. We only show the calculation of $sem_{Pr}(f, w)$, but $ver_{Pr}(f, w)$ can easily be derived from it.

$$
\begin{aligned}
& sem_{Pr}(f, w) \\
= & sem_{Pr}(\{i_f \oplus h_V \oplus \overline{h}(|\varphi|, \vec{V}) \oplus r'\}, w) \\
= & sem_{Pr}(\{i_f \oplus h_V \oplus \overline{h}(|\varphi|, \vec{V}) \oplus r'\}, \\
& \qquad \textbf{exhale } I \text{ ; } \textbf{havoc } \vec{V} \text{ ; } \textbf{inhale } I \text{ ; } \textbf{assume } \neg b) \\
= & sem_{Pr}(\{|i_f| \oplus h_V \oplus \overline{h}(|\varphi|, \vec{V}) \oplus r'\}, \textbf{havoc } \vec{V} \text{ ; } \textbf{inhale } I \text{ ; } \textbf{assume } \neg b) \\
= & sem_{Pr}(\{\overline{h}(|i_f| \oplus h_V \oplus \overline{h}(|\varphi|, \vec{V}) \oplus r', \vec{V})\} \oplus h(\vec{V}), \textbf{inhale } I \text{ ; } \textbf{assume } \neg b) \\
= & sem_{Pr}(\{\overline{h}(|i_f|, \vec{V})\} \oplus \{\overline{h}(|\varphi|, \vec{V})\} \oplus \{r'\} \oplus h(\vec{V}), \textbf{inhale } I \text{ ; } \textbf{assume } \neg b) \\
= & sem_{Pr}(\{\overline{h}(|i_f|, \vec{V})\} \oplus \{\overline{h}(|\varphi|, \vec{V})\} \oplus \{r'\} \oplus h(\vec{V}) \oplus Inh(I), \textbf{assume } \neg b) \\
= & sem_{Pr}(\{\overline{h}(|i_f|, \vec{V})\} \oplus F, \textbf{assume } \neg b) \\
>> & sem_{Pr}(F, \textbf{assume } \neg b) && \text{(because } mono_{Pr}(\textbf{assume } \neg b)\text{)} \\
= & filter_{\neg b}(F)
\end{aligned}
$$

4. $sem_{Pr}(F, inl^n_{Pr}(\vec{l}, w)) >> sem_{Pr}(F, w)$ and $ver_{Pr}(F, inl^n_{Pr}(\vec{l}, w))$:
   We apply $PSP^n_{Pr}(\vec{l}, w)$ to $A' := F$ and $A := F$. We know that $ver_{Pr}(F, w)$, so we just need to show $domain(F) \subseteq \vec{l}$ to satisfy the left hand side of $PSP^n_{Pr}(\vec{l}, w)$.

$$
\begin{aligned}
domain(F) &= domain(Inh(I)) \cup domain(h(\vec{V})) \cup \sigma(\overline{h}(|\varphi|, \vec{V})) \cup \sigma(r') \\
&= domain(Inh(I)) \cup \vec{V} \cup (\sigma(\varphi) - \vec{V}) \cup (\sigma(r) - \vec{V}) \\
&\subseteq domain(Inh(I)) \cup \vec{V} \cup (\sigma(\varphi)) \\
&\subseteq \sigma(\varphi) \\
&\subseteq l
\end{aligned}
$$

We then get the right hand side of $PSP^n_{Pr}(\vec{l}, w)$, which gives us the desired results.

Putting everything together:
Using $mono_{Pr}(inl^n_{Pr}(\vec{l}, w))$, we get $ver_{Pr}(\{\varphi'\}, inl^{n+1}_{Pr}(\vec{l}, w))$. We can now conclude:

$$
\begin{aligned}
& sem_{Pr}(\{\varphi'\}, inl^{n+1}_{Pr}(\vec{l}, w)) \\
=& sem_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s)\ ;\ inl^n_{Pr}(\vec{l}, w)) \\
=& sem_{Pr}(sem_{Pr}(\{\varphi'\}, inl^n_{Pr}(\vec{l}, s)), inl^n_{Pr}(\vec{l}, w)) \\
>>& sem_{Pr}(sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I)), s) \oplus \{r'\}, inl^n_{Pr}(\vec{l}, w)) \\
& \qquad \text{(using result 2 and } mono_{Pr}(inl^n_{Pr}(\vec{l}, w))) \\
>>& sem_{Pr}(filter_b(\{\overline{h}(|\varphi|, \vec{V})\} \oplus h(\vec{V}) \oplus Inh(I) \oplus \{r'\}), inl^n_{Pr}(\vec{l}, w)) \\
& \qquad \text{(using result 1 and } mono_{Pr}(inl^n_{Pr}(\vec{l}, w))) \\
=& sem_{Pr}(F, inl^n_{Pr}(\vec{l}, w)) \\
>>& sem_{Pr}(F, w) \qquad \text{(using result 4)} \\
>>& filter_{\neg b}(F) \qquad \text{(using result 3)} \\
=& sem_{Pr}(\{\varphi\}, w) \qquad \square
\end{aligned}
$$

Chapter 7

Completeness of Static Inlining

In Chapter 6 we proved that, under the soundness condition, we have soundness of inlining. That is, if an inlined program (up to any bound n) does not verify, we know for sure that the original program, with whatever annotation, does not verify either.[1]

However, the soundness theorem goes in only one direction: It does not tell us anything about the situation where the inlined program verifies, and therefore does not inform us on how close the relationship is between the behaviour of the inlined program and the original program. As an example, we could define the following inlining function:

$$inl^n_{Pr}(l, s) := \textbf{skip}$$

This inlining function is sound. Indeed, it always verifies, thus it is never wrong when it finds a fundamental bug (since it never finds any).

We therefore want a completeness property for the inlining function we have defined. The ideal completeness property would state that, under certain conditions, if the inlined program verifies, then there exists an annotation for the program (loop invariants, method preconditions and postconditions) such that this annotated program verifies. Such a property would indeed show us that our inlining function is a good model for what happens with the annotated program.

This completeness property does not hold, for one obvious reason: Inlining always only goes up to a bounded depth, so it cannot provide any information on executions which go deeper than this bound. Moreover, while loops and recursive methods are in general statically unbounded. Therefore, for any bound, we can find an example of incompleteness of static inlining, where the inlined program up to this bound verifies, whereas no annotation would make the original program verify.

[1] Assuming completeness of the verifier.

In this chapter, we try to understand which conditions are required to get a completeness result. We then sketch how to prove such a completeness property. Intuitively, the inlined program behaves as if all loop iterations and method bodies would perfectly know the whole context. Therefore, when the inlined program verifies, we try to construct annotations which describe the whole context, in the hope that this reproduces the behavior of the inlined program, and thus makes the original annotated program verify. These annotations are based on the idea of a strongest postcondition, that is an assertion which describes exactly the program state after the execution of a statement from an initial empty state.

The structure of the chapter is as follows. We first define the syntactic transformation of "bounding" a program up to a bound, to align the execution of the original program on the one of the inlined program. The completeness property we study is with respect to this bounded program: If the inlined program verifies, does there exist an annotation which makes the bounded program verify? We show with two examples that this completeness property does not hold in general in Viper. One example is based on the lack of indices in loops and in recursive methods, and the other one is based on restrictions on how permissions can be used in Viper. We finally express this completeness property in our general framework, and sketch a proof for it, under conditions which disallow the previous examples, and with an assumption that there exists a strongest postcondition.

The concepts presented in this chapter have not been fully formalized yet. We therefore stay at an informal level, in order to convey the intuition behind it.

7.1 Syntactic transformation: bounded program

As explained, we need to modify the original program, such that the execution stops if it reaches a depth which cannot be reached in the inlined program. This way, we can say something about the original program based on information from the inlined program. We define a syntactic transformation, $bound^n(s, Pr)$, which takes as input an initial statement s and a program Pr, and outputs another initial statement and another program, as illustrated in Figure 7.1. Even though $bound^n(s, Pr)$ is a syntactic transformation, it represents the same program semantically, except that the execution is sometimes stopped.

Listing 7.1 shows a simple program (with dots representing some statements), consisting of an initial statement with one loop and calling one method $m1$. The method $m1$ calls another method $m2$. Listing 7.2 shows the result of the syntactic transformation $bound^{42}$ to this program. We intrument the program in two ways:

Listing 7.1: Original program.

```
method initial() {
    ...
    m1(...)
    ...
    while (b)
    {
        ...
    }
    ...
}

method m1(...) {
    ...
    m2(...)
    ...
}
```

Listing 7.2: Bounded program.

```
method initial() {
    var depth: Int := 42
    ...
    assume depth > 0
    m1(..., depth - 1)
    ...
    var before_depth: Int := depth
    while (b && depth > 0)
    {
        depth := depth - 1
        ...
    }
    assume !b
    depth := before_loop_depth
    ...
}

method m1(..., old_depth: Int) {
    var depth: Int := old_d
    assume depth > 0
    ...
    m2(..., depth - 1)
    ...
}
```

Figure 7.1: An example of the syntactic transformation $bound^{42}(s, Pr)$.

1. We keep track of the current depth of the execution, via the variable *depth*.
2. We stop the execution if *depth* is zero or less, similarly to what happens with the inline function.

The current depth is transmitted as an additional argument to methods, which then assign it to the variable *depth* to be allowed to modify it. Every method call decrements the current depth by one.

We do the same for loops. Before every loop, we record the current depth, using an auxiliary variable. Moreover, we add the condition $depth > 0$ to the loop condition. This ensures that we do not enter the loop if we reached the bound. We assume $!b$ after the loop, to stop executions which should have enter the loop one more, but which did not because of the condition $depth > 0$. Furthermore, at every iteration of the loop, we decrement the depth by one. Eventually, we go back to the depth before the loop.

Using this syntactic transformation, we can now express the completeness

property we want to have and prove:

Definition 7.1 ***Completeness property***
If

$$inl^n_{Pr}(modif(s), s)$$

verifies, then there exists an annotation A such that

$$bound^n(s, annotate(Pr, A))$$

verifies.

This completeness property does not hold in Viper, as we show in the next chapter. Similarly to what we did in Chapter 3 for the soundness property, we try to find a *completeness condition* such that this holds.

Since this chapter is not fully formalized, it is not clear whether this will be the definitive shape of the completeness. In particular, it could make more sense to define a *semantic* notion of a bounded program, instead of a *syntactic* one.

7.2 Examples of incompleteness in Viper

We have now expressed the completeness property we want to have. Like the soundness property, it does not hold in general. We show, in this section, two cases in Viper in which this completeness does not hold. The first case occurs when loops or recursive methods do not have an index to use in the annotations. This case is a general one, and is relevant for our general framework. The second one occurs because of a permission gap in Viper, and is therefore restricted to Viper.

7.2.1 Loops and recursive methods without index

We show the first example on Listing 7.3 and Listing 7.4. Listing 7.3 shows the bounded program (with a bound of 2), whereas Listing 7.4 shows the inlined program (with the same bound of 2).

We use two abstract methods (methods without any specified body), *flip_coin* and *thow_dice*, to encode havocing (which currently does not exist in Viper). *flip_coin* simply returns a boolean with no annotations, thereby havocing a boolean. *throw_dice* havocs an integer, and contrains it to be between 1 and 6 included.

This program does something simple. It keeps track of a counter c, initialized at 0. It then flips a coin b, and if b is true, it throws a dice (result between 1 and 6 included), and adds the result of this dice to the counter. It then goes back to flipping a coin and does the same, until b is false.

Listing 7.3: Bounded program (bound of 2).

```
method flip_coin() returns (b: Bool)

method throw_dice() returns (i: Int)
        ensures i >= 1 && i <= 6

method initial_method() {
    var depth: Int := 2
    var b: Bool
    var d: Int
    var c: Int := 0
    b := flip_coin()
    var before_depth: Int := depth
    while (b && depth > 0) {
        depth := depth - 1
        d := throw_dice()
        c := c + d
        b := flip_coin()
    }
    assume !b
    assert c <= 12
}
```

Listing 7.4: Inlined program (bound of 2).

```
method flip_coin() returns (b: Bool)

method throw_dice() returns (i: Int)
        ensures i >= 1 && i <= 6

method initial_method() {
    var b: Bool
    var d: Int
    var c: Int := 0
    b := flip_coin()
    if (b) {
        d := throw_dice()
        c := c + d
        b := flip_coin()
        if (b) {
            d := throw_dice()
            c := c + d
            b := flip_coin()
            assume !b
        }
    }
    assume !b
    assert c <= 12
}
```

The inlined program verifies. Indeed, the assertion **assert** c <= 12 (line 23) holds since we do at most two iterations. However, if we do not use *depth* in a loop invariant, then there is no way to write a loop invariant which would make the same assertion true in the bounded program, on line 20. Indeed, we only know that $c = 0$ before the first assertion, c is between 1 and 6 after the first iteration, and between 2 and 12 after the second one. Since we do not have any index (except *depth*) for knowing in which iteration we are, it is not possible to write the intermediate assertion saying that c is between 1 and 6.

This example shows, therefore, that we need indices to be able to speak about a particular iteration. Indeed, the annotation A we want in our completeness property cannot refer to the variable *depth*. The same reasoning also applies for recursive methods, for which we also need indices.

7.2.2 Permission gap

The lack of indices is not the only cause of incompleteness. Another cause, at least in Viper, is a gap in describing permissions.

Take the example shown on Listing 7.5 and Listing 7.6. Listing 7.5 shows the original annotated program. The purpose of lines 4 to 6 is to get some permission to $x.f$ which is strictly greater than zero, and less than half, but for which there does not exist any lower bound strictly greater than zero. More precisely, after line 6, we know that $\exists n.n >= 2 \wedge perm(x.f) == \frac{1}{n}$. In particular, $perm(x.f) > 0$ and $perm(x.f) <= 1/2$.

It is impossible to perfectly describe this permission amount in the precondition of *callee*, since we cannot refer to n. We then call the function *callee*, and want to assert that the permission held to $x.f$ is not zero, and less than half, which would be true with the same permission amount as before. The question boils down to what should be the precondition and postcondition of the method *callee*.

callee is a simple method, which only asserts something that is almost a tautology. Indeed, as long as we have *some* permission to *x.f*, this assertion verifies. Therefore, we just need *some* permission in the precondition. However, we do not have a lower bound on the amount of permission we can transmit to *callee*. Moreover, we cannot use n in our precondition. Therefore, any possible precondition to make the method *callee* verify has to use a **wildcard**. **wildcard** in Viper means *some* permission amount, strictly between 0 and 1.[2] To make line 8 verify, we need at least *some* permission amount, which is also less than half, but we do not know how much permission we have in *callee*.

[2] As described in [1], this definition corresponds to inhaling a **wildcard**. Exhaling a **wildcard** results in exhaling some permission amount, strictly between 0 and the current permission held to the relevant location.

Listing 7.5: Original annotated program.

```
1  field f: Int
2
3  method initial_method(x: Ref) {
4      var n: Int
5      assume n >= 2
6      inhale acc(x.f, 1/n)
7      callee(x)
8      assert perm(x.f) > none && perm(x.f) <= 1/2
9  }
10
11 method callee(x: Ref)
12     requires acc(x.f, wildcard)
13     ensures acc(x.f, wildcard)
14 {
15     assert x.f < 0 || x.f >= 0
16 }
```

Listing 7.6: Inlined program (bound of 1)

```
1  field f: Int
2
3  method initial_method_inlined(x: Ref) {
4      var n: Int
5      assume n >= 2
6      inhale acc(x.f, 1/n)
7      assert x.f < 0 || x.f >= 0
8      assert perm(x.f) > none && perm(x.f) <= 1/2
9  }
```

We therefore can only use a **wildcard** to give back *some* permission to the initial method. However, even with this annotation for the *callee* method, line 8 cannot verify. Indeed, by using **wildcard**, we forget the precise amount of permission to $x.f$. In particular, we lost the information that this permission amount is less than half. Therefore, there does not exist any annotation for *callee* such that this program verifies.

On the other hand, the inlined program shown on Listing 7.6 verifies, the reason being that the amount of permission is not forgotten by the transmission to a method. This is, therefore, another example of incompleteness in Viper.

There exist several possibilities to bridge this gap:

1. Instrument the program with ghost variables, to record the permission amounts transmitted. These variables would be transmitted to the methods via their arguments, and would be assigned using the **perm** feature.

Listing 7.7: Original program.

```
1  field f: Int
2
3  method test(x: Ref) {
4      inhale acc(x.f, wildcard)
5      var p: Perm
6      var i: Int := 0
7      while (i < 1)
8          invariant ...
9      {
10         p := perm(x.f)
11         i := i + 1
12     }
13     assert p == perm(x.f)
14 }
```

Listing 7.8: Inlined program (bound of 1).

```
1  field f: Int
2
3  method inlined(x: Ref) {
4      inhale acc(x.f, wildcard)
5      var p: Perm
6      var i: Int := 0
7      if (i < 1)
8      {
9          p := perm(x.f)
10         i := i + 1
11         assume !(i < 1)
12     }
13     assert p == perm(x.f)
14 }
```

Figure 7.2: Existential permissions require the soundness condition.

2. Use abstract read permissions [9, 22].
3. Add support for existential permissions in annotations. This would mean, for example, accept the use of syntactic assertions such as $\exists n.n \geq 2 \wedge acc(x.f, 1/n)$ as annotations. This would require giving a semantics to inhaling and exhaling assertions with existential permissions, and extending the definition of *supported* assertions, since assertions with existential permissions would not be supported in general.

Remark 7.2 *The third solution, existential permissions, is not sufficient to get the completeness property, as we show in Figure 7.2. Even with existential permissions, one cannot write an invariant such that line 13 of the original program (Listing 7.7) verifies. However, the inlined program shown on Listing 7.8 verifies. A reason is that the body of the loop is not* ***framing*** *(and not even* ***mono****). It seems, however, that it is sufficient to prove the completeness property under the soundness condition. This is not a problem, since the purpose of this completeness result is to show that inlining under the soundness condition is a good approximation of what happens in the original program.*

It is not clear which solution is the best. However, in Chapter 8, we tend towards the third solution for Viper.

7.3 Sketch of completeness

We have seen two examples of incompleteness in Viper. We now explore completeness, in this section, for programs where we do not have these issues. Since we have not proven any completeness result yet, we only

show a possible result of completeness, with a sketch of a proof. This result is based on an assumption regarding the strongest postcondition: That is, we assume that we can take a statement and compute its strongest postcondition. We then use this assumption to describe how we would construct an annotation for the program, and why the bounded program verifies using this annotation.

The previous sections were mainly about Viper. In this section, we consider completeness in our general framework.

7.3.1 Strongest postcondition assumption

The *strongest postcondition* assumption is formally defined as follows:

Definition 7.3 ***Strongest postcondition assumption***

$$\exists SP.\, (\forall s.ver_{Pr}(\{u\}, s) \implies sem_{Pr}(\{u\}, s) = Inh(SP(s)))$$

This assumes the existence of a function SP which takes a statement as input, and outputs a semantic assertion, which is similar to a strongest postcondition (hence the name SP) in the following sense: Executing the statement s from the empty state (under the condition that it verifies) gives the minimal satisfying set of $SP(s)$, that is $Inh(SP(s))$. Therefore, executing s from an empty state is almost equivalent to inhaling $SP(s)$, that is

$$s \approx \textbf{inhale}\ SP(s)$$

These two statements are in general not equivalent in terms of semantics. Indeed, take $s = \textbf{var}\ x.\ s$ declares the variable x, something which inhaling an assertion cannot do. Indeed, the semantics of **inhale** P (for a well-formed assertion) is defined for an initial state φ only if

$$\forall i \in Inh(P).\sigma(i) \subseteq \sigma(\varphi)$$

This means, in particular, that **inhale** cannot define new variables. Hence, **inhale** $SP(s)$ is not in general equivalent to s. This is why we express this assumption using $Inh(SP(s))$, and not **inhale** $SP(s)$.

This strongest postcondition assumption does not hold in general in Viper, because of the permission gap. However, if we bridge this gap, it seems that such a statement is provable. We discuss how to prove this assumption for Viper in Section 8.6.

7.3.2 Constructing the annotation

The idea is the following. The strongest postcondition function, SP, returns an assertion which essentially captures all the information available in the

inlined program. Therefore, adding annotations constructed with this SP function should provide the bounded program with the information used by the inlined program, thus making the verification of the bounded program equivalent to the verification of the inlined program.

We show through two examples, in Figure 7.3, how to construct an annotation such that the bounded program verifies, using the strongest postcondition assumption. We assume the strongest postcondition assumption, and we assume that all loops and method calls are indexed: That is, there is a way to uniquely refer to one instance of a method call or a specific loop iteration in an assertion. The idea is to construct annotations which correspond to the beginning and end of every method call, and the beginning and end of every loop iteration, in the inlined program.

The first example, Listing 7.9, deals with constructing annotations for a method. The inlined version of this program is shown on Listing 7.10, where sy corresponds to sx renamed with y instead of x, and sz corresponds to sx renamed with z instead of x. We have a simple method *callee*, with two arguments, an integer i and a variable x of type T. We use i as an index, considering in this case that $i = 0$ during the first method call, and that $i = 1$ during the second method call.

We construct a precondition and a postcondition for each method call, two of each in this case. We use our SP function on the inlined program to do that. Before the first method call, $i = 0$, and only $s1$ has been executed. We therefore add the precondition $i == 0 ==> SP(s1)$. This precondition should be satisfied before the method call, since the inlined program verifies until this point. The renamed method body, sy should then verify, and the set of states afterwards should be $SP(s1; sy)$, as hinted on the inlined program. We therefore add the postcondition $i == 0 ==> SP(s1; sy)$ to the method. We do the same for the second method call, as shown on the figure (red before the method call, and black afterwards), which gives us the precondition $i == 1 ==> SP(s1; sy; s2)$ and the postcondition $i == 1 ==> SP(s1; sy; s2; sz)$.

Writing several preconditions (resp. postconditions) in Viper has the effect of conjoining them. Our two preconditions (resp. postconditions) are mutually exclusive (since i is either 0 or 1). Therefore, our reasoning for the first method call should also work for the second one.

We construct loop invariants similarly, as illustrated on Listing 7.11 (annotated program) and Listing 7.12 (inlined program). We assume that i is an index for the loop, incremented at every iteration, and starting at 0. As illustrated in blue in Figure 7.3, we first write a loop invariant which holds when entering the loop, $i == 0 ==> SP(s0)$. We then construct loop invariants which have to hold after each iteration, taking into account one more execution each

Listing 7.9: Example for a method.

```
1  method initial_method() {
2      s1
3      callee(i, y)
4      s2
5      callee(i, z)
6      s3
7  }
8
9  method callee(i: Int, x: T)
10     requires i == 0 ==> SP(s1)
11     requires i == 1 ==> SP(s1 ; sy ; s2)
12     ensures i == 0 ==> SP(s1 ; sy )
13     ensures i == 1 ==> SP(s1 ; sy ; s2 ; sz )
14 {
15     sx
16 }
```

Listing 7.10: Inlined.

```
method inlined() {
    s1          i = 0
    sy          i = 0
    s2          i = 1
    sz          i = 1
    s3
}
```

Listing 7.11: Example for a loop.

```
1  method initial_method(n: Int) {
2      s0
3      while (i < n)
4          invariant i == 0 ==> SP(s0)
5          invariant i == 1 ==> SP(s0 ; s)
6          invariant i == 2 ==> SP(s0 ; s ; s)
7          ...
8      {
9          s
10     }
11     ...
12 }
```

Listing 7.12: Inlined.

```
method inlined(n: Int) {
    s0                  i = 0
    if (i < n) {
        s               i = 1
        if (i < n) {
            s           i = 2
            ...
        }
    }
    ...
}
```

Figure 7.3: Illustration of how to construct annotations using the strongest postcondition assumption.

time, such as $i == 1 ==> SP(s0; s)$, $i == 2 ==> SP(s0; s; s)$, ... We stop when we reach the inlining bound.

This is the current state of completeness. The formalization of everything presented here is planned in the future.

There still remains one gap between the examples of incompleteness presented earlier, and the proof as sketched here. For this proof, we assume that every method call is indexed, whereas examples of incompleteness we can find (for the moment) are only about recursive methods which are not indexed. We therefore need to continue investigate whether it is possible to write annotations in the case of several method calls without indices, or exhibit a counter-example.

We have now proved a soundness result in our framework, and sketched a completeness result assuming the existence of a *strongest postcondition* function. The next chapter describes how to define the parameters for instantiating our general framework to get a model of the Viper language, and also discusses (with a sketch of a proof) the strongest postcondition assumption in Viper.

Chapter 8

Instantiating the Parametric Language with Viper

Chapter 4 and 5 define a general model of a language, whose semantics is built on a separation algebra. In Chapter 6, we expressed and proved the soundness of inlining under a soundness condition in this language. Moreover, we have informally expressed a completeness property for inlining in Chapter 7, and sketched a proof for it. By inputting the right parameters, this general framework can be instantiated as a model for concrete verification languages. Furthermore, this parametric language is strongly inspired by the Viper language. This chapter shows how to instantiate our framework to get a model for a simplified version of the Viper language.

Section 8.1 defines the simplified version of Viper we consider. Section 8.2 defines the separation algebra and the store interface for this simplified version of Viper. Section 8.3 discusses assertions. Section 8.4 shows an example of how to define a rename interface which satisfies all requirements. The core of this rename interface has been formalized and proved in Isabelle/HOL (see Appendix A). Section 8.5 discusses custom statements and the custom interface. Section 8.6 sketches how to prove the strongest postcondition assumption, to leverage the completeness of inlining result from Chapter 7. Finally, Section 8.7 briefly discusses how to use the theoretical results from the parametric language and the instantiation of this model for the Viper language to create two useful features for Viper: The early error detection feature described in the introduction, and a feature to speed up re-verification using caching.

8.1 A simplified version of Viper

There are several reasons for working in this chapter with a simplified version of Viper. First, Viper does not have clearly defined semantics. Viper is a

powerful intermediate language for verification, with powerful features such as functions and predicates, and giving a formal semantics for these features is beyond the scope of this thesis. Second, the core of this thesis is the parametric language with the theoretical results. This parametric language with its requirements will be modified in the future, thus this chapter is more a proof-of-concept than a real instantiation.

In this simplified version, we do not permit referring to previous states, we do not accept inhale-exhale assertions in annotations, we do not consider typing for variables, and we ignore powerful features such as functions, predicates and magic wands.

8.1.1 Disallowing references to previous states

As explained in the background chapter (Section 2.2.2), Viper semantics can be defined as a trace semantics, to enable referring to past heaps. This is mostly done through two features. Labels, which record the heap at a certain point of the execution, and old expressions, with or without a label, referring to the heaps of past states. In this simplified version, we do not allow the use of labels and old-expressions.

However, it seems possible to keep track of a trace in a state, by adding a fourth component. This fourth component would be a finite mapping from labels to states. Moreover, two states could be added if they agree on the states which are mapped from the same labels. The sum of two states would simply be the mapping obtained by combining both. This idea has to be explored more, to know if this is sufficient, and in general to know what happens with assertions, and the **mono** and **framing** properties.

8.1.2 Inhale-exhale assertions

As explained in [1], inhale-exhale assertions $[A, B]$ are assertions which behave like A when inhaled, and like B when exhaled. These assertions can be used in a postcondition to encode a leak check, or to enhance verification by using properties justified externally [15]. It is quite clear that inhale-exhale assertions cannot be encoded as semantic assertions with the same meaning. We therefore disallow the use of inhale-exhale assertions.

8.1.3 Types of variables

Our parametric language does not consider types of variables. Indeed, we define everything as if we had only type of variable.

Types of variables could be encoded in the store of a state, adding the type to the mapping. They could otherwise be checked as part of the *wfProg* function, which checks whether a program is well-formed. On top of that, we would

have to redefine the meaning of the h function, used in the semantics of **var** and **havoc**. We would also have to add a type of types in our parameters, and add a type when declaring variables with a **var** statement. This could be done in a future work, but is currently disallowed.

8.1.4 Other features

As explained at the beginning of this section, predicates, magic wands and functions are examples of powerful features which lack clearly-defined semantics. This is why we currently disallow these features.

However, it seems that (recursive and non-recursive) predicates and magic wands could be encoded as part of another permission mask, and would not complicate the instantiation too much. This could be explored in the future.

8.2 Separation algebra, variables and store

We now define our separation algebra, which captures Viper's state model.

Definition 8.1 ***Set of Viper states*** Σ***, store*** σ ***and impure part*** C

Let V be the set of all strings (names of variables), Val be the set of possibles values for variables (at least integers, rationals and booleans), $Addr$ be a set of integers (addresses in the heap), and $Field$ be the set of fields defined by the program (strings).

We define the following types:

$$\begin{aligned} Permissions &:= \mathcal{Q} \cap [0,1] \\ Stores &:= V \rightharpoonup_{fin} Val \\ PermissionMasks &:= Addr \times Field \to Permission \\ Heaps &:= Addr \times Field \rightharpoonup_{fin} Val \end{aligned}$$

Σ *is the set of all states* $(s, \pi, h) \in Stores \times PermissionMasks \times Heaps$ *such that*

1. $\{(o,f) | \pi(o,f) > 0\}$ *is finite.*
2. *The heap h is only defined on* $\{(o,f) | \pi(o,f) > 0\}$.

Let domain be the function giving the domain of a mapping. The store of a state is

$$\sigma((s,\pi,h)) := domain(s)$$

Let $s_\varnothing$ be the empty store (empty mapping). The impure part of a state is then

$$C((s,\pi,h)) := (s_\varnothing, \pi, h)$$

Definition 8.2 ***Combination of stores***
Let domain be the function giving the domain of a mapping. Two stores s_1 and s_2 can be combined, written $s_1 \# s_2$, if and only if

$$\forall x \in domain(s_1) \cap domain(s_2).s_1(x) = s_2(x)$$

In this case, the combination of two stores is then defined as

$$domain(s_1 \oplus s_2) = domain(s_1) \cup domain(s_2)$$
$$\forall x \in domain(s_1 \oplus s_2).(s_1 \oplus s_2)(x) = \begin{cases} s_1(x) & \text{if } x \in domain(s_1) \\ s_2(x) & \text{otherwise} \end{cases}$$

Definition 8.3 ***Combination of permission masks***
Two permission masks π_1 and π_2 can be added, written $\pi_1 \# \pi_2$, if and only if

$$\forall (o, f) \in Addr \times Field.\pi_1(o, f) + \pi_2(o, f) \leq 1$$

In this case,

$$\forall (o, f) \in Addr \times Field.(\pi_1 \oplus \pi_2)(o, f) = \pi_1(o, f) + \pi_2(o, f)$$

Definition 8.4 ***Combination of heaps***
Let domain be the function giving the domain of a mapping. Two heaps h_1 and h_2 can be added, written $h_1 \# h_2$, if and only

$$\forall (o, f) \in domain(h_1) \cap domain(h_2).h_1(o, f) = h_2(o, f)$$

In this case, the combination of two heaps is then defined as

$$domain(h_1 \oplus h_2) = domain(h_1) \cup domain(h_2)$$
$$\forall (o, f) \in domain(h_1 \oplus h_2).(h_1 \oplus h_2)(o, f) = \begin{cases} h_1(o, f) & \text{if } (o, f) \in domain(h_1) \\ h_2(o, f) & \text{otherwise} \end{cases}$$

Definition 8.5 ***Addition of states $\oplus$ and neutral element u***
Two states (s_1, π_1, h_1) and (s_2, π_2, h_2) can be added if and only if

$$(s_1, \pi_1, h_1) \# (s_2, \pi_2, h_2) \iff s_1 \# s_2 \wedge \pi_1 \# \pi_2 \wedge h_1 \# h_2$$

In this case, the sum of two states is defined as

$$(s_1, \pi_1, h_1) \oplus (s_2, \pi_2, h_2) = (s_1 \oplus s_2, \pi_1 \oplus \pi_2, h_1 \oplus h_2)$$

Let $s_\varnothing$ represent the empty store, that is an empty mapping. Let $h_\varnothing$ represent the empty heap, that is also an empty mapping. Let π_0 be the zero mask, that is

$$\forall (o, f) \in Addr \times Fields.\pi_0(o, f) = 0$$

The neutral state u is then defined as

$$u := (s_\varnothing, \pi_0, h_\varnothing)$$

All requirements described in Chapter 4 can be proved from these definitions. In particular, the core of a state corresponds exactly to the store:

Lemma 8.6 ***Core of a state***

$$|(s, \pi, h)| = (s, \pi_0, h_\varnothing)$$

8.3 Assertions

As we already discussed in Section 5.3.3, there is a mismatch between syntactic assertions from Viper and semantic assertions. In this section, we do not dive deep into syntactic assertions. We first sketch a way for defining an encoding from syntactic to semantic assertions, and how to define the *wellDefinedAssert* function. We then briefly discuss why syntactic annotations are encoded into supported and intuitionistic semantic assertions.

8.3.1 From syntactic to semantic assertions

As we saw, different syntactic assertions in Viper must be encoded into the same semantic assertion. Let us denote *SynAssert* the set of all syntactic assertions one can write in Viper.

We encode syntactic assertions into semantic ones using the following *makeSemantic* function:

Definition 8.7 ***Encoding a syntactic assertion into a semantic one***
The function makeSemantic takes an element from SynAssert and returns the semantic assertion such that

$$\mathit{makeSemantic}(A)(\varphi) := \begin{cases} \top & \text{if } A \text{ is well-defined and true in } \varphi \\ \bot & \text{otherwise} \end{cases}$$

That is, the corresponding semantic assertion is true when the syntactic assertion is both well-defined and true, and false otherwise. For example, this means that we encode the syntactic assertion **`false`** and `1 / 0 == 1` into the same semantic assertion, the one which is always false, hence

$$\mathit{makeSemantic}(\texttt{1/0 == 1}) = \mathit{makeSemantic}(\textbf{\texttt{false}})$$

Another example is

$$\mathit{makeSemantic}(\texttt{x.f == x.f}) = \mathit{makeSemantic}(\textbf{\texttt{perm}}\texttt{(x.f) > 0})$$

Indeed, `x.f == x.f` is always true when it is defined, but it is defined only when some permission amount strictly greater than zero is held to the heap location $x.f$.

We define the *wellDefinedAssert* function, which encodes whether a semantic assertion is well-defined in a state, as follows:

Definition 8.8 ***Well-defined assertion***

$$\begin{aligned} &\mathit{wellDefinedAssert}(P, \varphi) \\ \iff &(\exists A \in \mathit{SynAssert}.\ \mathit{makeSemantic}(A) = P \wedge A \text{ is well-defined in state } \varphi) \end{aligned}$$

This definition means that if it is possible to rewrite the syntactic assertion A into A' such that both assertions are encoded into the same semantic assertion, and one of them is well-defined in the state φ, then their common encoding into a semantic assertion is also well-defined in the state φ. In a way, it acts as if any syntactic assertion is rewritten into a minimal syntactic assertion which is equivalent. As an example, the semantic assertion corresponding to `x.f == x.f` is always well-defined, since `perm(x.f) > 0` is always well-defined.

As discussed in Section 5.3.3, there is an inherent mismatch in trying to encode syntactic assertions into semantic ones. This definition is therefore not perfect. A consequence of this definition is that if we have well-definedness in Viper, we have well-definedness in the parametric language. Another one is that if we define a function which returns the set of variables read by a syntactic assertion *readSynAssert*, we have

$$\mathit{readAssert}(\mathit{makeSemantic}(A)) \subseteq \mathit{readSynAssert}(A)$$

8.3.2 Annotations are supported and intuitionistic

We only give a high view of why annotations are encoded into well-formed semantic assertions, since a formal proof would require to first define syntactic assertions, and then to formally define the meaning of self-framing assertions (following [20]).

First, annotations cannot use the keywords **perm** and **forperm**. Moreover, Viper disallows implications with accessibility predicates on the left side. For example, the assertion `acc(x.f) ==> ...` is not valid in Viper. Using these restrictions, it seems possible to show that Viper annotations are encoded into intuitionistic semantic assertions.

Furthermore, annotations are self-framing. As explained in the background chapter (Section 2.2.1), a self-framing assertion is an assertion which carries at least permissions for the locations it reads. Moreover, in Viper, it is not possible to write a disjunction with accessibility predicates. For example, the syntactic assertion `acc(x.f) || acc(y.f)` is not allowed in Viper. The corresponding semantic assertion would not be supported.

Let us describe a bit more why a self-framing assertion in Viper is supported, and why assertions which are not self-framing can be not supported. Consider $P :=$ `acc(x.f, 1/2) && x.f == 5` and $Q :=$ `x.f == 5`. P is self-framing, whereas Q is not. P is encoded into a supported assertion. Indeed, any state satisfying P has at least half permission to $x.f$, and $x.f == 5$. Therefore, all states have the same support, the state φ which has only half permission to $x.f$, and where $x.f == 5$. A state satisfies P if and only if it is stronger than φ. On the other hand, Q is encoded into an assertion which is not supported. Indeed, a state $(s, \pi, h) \in \Sigma$ satisfies Q if and only if $h(x, f) = 5$. This implies $\pi(x, f) > 0$. However, one cannot find a support for this state, since this support must satisfy $\pi(x, f) > 0$, and there does not exist any lower bound for $\pi(x, f)$ other than zero.

These restrictions on Viper annotations therefore imply that they are encoded into well-formed (supported and intuitionistic) semantic assertions. This result should be formally proved in the future. This means that the semantics of **`inhale`** and **`exhale`** statements in Viper have to match the ones from the parametric language only for annotations which are encoded into well-formed semantic assertions. We describe the behavior of **`inhale`** and **`exhale`** for other kinds of assertions in the custom inferface, in Section 8.5.

8.4 Rename interface

We show in this section one way of instantiating the rename interface, which satisfies all requirements. Viper variables are strings, and strings are countable, they are therefore in bijection with natural numbers. To simplify the formalization, we base our renaming framework on natural numbers. It is easy to transform this into a renaming interface based on strings, using a bijection between strings and numbers.

We first define two ways of renaming a natural number into another, depending on the renaming quadruple. We then show how to invert a renaming quadruple. We finally describe how to lift the function renaming a natural number to construct the different functions of the rename interface. The core part of this section, namely the renaming of natural numbers and the inversion of a renaming quadruple, have been formalized in Isabelle/HOL (see Appendix A), with the corresponding results (Requirement 5.37).

8.4.1 Renaming an element

A renaming quadruple is a quadruple of lists of natural numbers

$$(\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain})$$

Elements of $\overrightarrow{old}$ should be replaced by the corresponding elements of $\overrightarrow{new}$, and no element from $\overrightarrow{domain}$ should be renamed into an element from $\overrightarrow{avoid}$.

It is well-formed if and only if $\overrightarrow{old}$ and $\overrightarrow{new}$ have the same length, elements of $\overrightarrow{old}$ are distinct, and elements of $\overrightarrow{new}$ are distinct.

We distinguish two cases for renaming. The first case is a really special case we mostly use for inverting a renaming quadruple. The second one is a general case.

First case: old and new correspond to the same set, avoid is empty

This case is a special case, whose only purpose is to make it simple to invert a renaming quadruple. We only use this case when there is no number to avoid in the domain ($\overrightarrow{avoid} = []$), and when $\overrightarrow{old}$ and $\overrightarrow{new}$ contain exactly the same set of numbers. Renaming in this case is simple:

Definition 8.9 ***Renaming a number (first case)***

$$rename_1((\overrightarrow{old}, \overrightarrow{new}, [], \overrightarrow{domain}))(x) := \begin{cases} \overrightarrow{new}[i] & \text{if } \exists i < length(\overrightarrow{old}).x = \overrightarrow{old}[i] \\ x & \text{otherwise} \end{cases}$$

If $x \in \overrightarrow{old}$, then we rename it to its corresponding element in $\overrightarrow{new}$ (we do not have any choice). In the other case, this renaming function is the identity. $rename_1$ is clearly a bijection, and fulfills the requirements (avoids what there is to avoid, namely nothing).

Second case: General case

This general case is less simple. We need to avoid renaming numbers from $\overrightarrow{domain}$ into numbers from $\overrightarrow{avoid}$, and we need to make sure our renaming is bijective.

This renaming function in the general case is built on the following lemma:

Lemma 8.10 *Let $\vec{a}$ and $\vec{b}$ be two lists of numbers, such that*

1. $length(\vec{a}) = length(\vec{b})$.
2. *Elements of $\vec{a}$ are distinct.*
3. *Elements of $\vec{b}$ are distinct.*

Then there exists a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that

1. *f is a bijection.*
2. $\forall i \in \mathbb{N}.i < length(\vec{a}) \implies f(\vec{a}[i]) = \vec{b}[i]$
3. $\exists M \in \mathbb{N}.\forall x \geq M.f(x) = x$

That is, with a well-formed renaming quadruple, we can find a bijective function f which maps $\overrightarrow{old}$ to $\overrightarrow{new}$ and which behaves like the identity function above a certain threshold.

We can now define the general case of renaming a number. $sum(\vec{l})$ denotes the sum of all elements of the list $\vec{l}$.

Definition 8.11 ***Renaming a number (second case)***
Let $t := (\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain})$ be a well-formed renaming quadruple. Let $\vec{a}$ be a list of distinct natural numbers, containing exactly the numbers from $\overrightarrow{old}$ and $\overrightarrow{domain}$. Let $\vec{b}$ be the list such that

1. $length(\vec{a}) = length(\vec{b})$
2. $\forall i, j \in \mathbb{N}. i < length(\overrightarrow{old}) \wedge j < length(\vec{a}) \wedge \vec{a}[j] = \overrightarrow{old}[i] \Rightarrow \vec{b}[j] = \overrightarrow{new}[i]$
3. $\forall j \in \mathbb{N}. j < length(\vec{a}) \wedge \vec{a}[j] \notin \overrightarrow{old} \Rightarrow \vec{b}[j] = \vec{a}[j] + sum(\overrightarrow{new}) + sum(\overrightarrow{avoid}) + 1$

It follows that elements of $\vec{b}$ are also distinct. Let f be a function as described in Lemma 8.10 for $\vec{a}$ and $\vec{b}$. Then

$$rename_2((\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain}), x) := f(x)$$

$rename_2$ is a bijection, thanks to Lemma 8.10. Moreover, it maps elements of $\overrightarrow{old}$ to elements of $\overrightarrow{new}$. Finally, it maps elements from $\overrightarrow{domain}$ to elements which are not in $\overrightarrow{avoid}$. Indeed, since we use natural numbers and we shift elements of $\overrightarrow{domain}$ not in $\overrightarrow{old}$ by adding $sum(\overrightarrow{new}) + sum(\overrightarrow{avoid}) + 1$ to them, it follows that the shifted element are greater than all elements in $\overrightarrow{new}$ and than all elements in $\overrightarrow{avoid}$. They therefore cannot be in $\overrightarrow{new}$ or in $\overrightarrow{avoid}$.

Another important property of this renaming function $rename_2$ is that it is equivalent to the identity function after a threshold, which makes it easy to invert.

Combining both renaming functions into one

Our rename function is defined as follows:

Definition 8.12 ***Renaming a number***

$$rename((\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain}), x) := \begin{cases} rename_1((\overrightarrow{old}, \overrightarrow{new}, [], \overrightarrow{domain}), x) \\ \quad \left(if\ \overrightarrow{avoid} = [] \wedge set(\overrightarrow{old}) = set(\overrightarrow{new})\right) \\ rename_2((\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain}), x) \\ \quad (otherwise) \end{cases}$$

8.4.2 Inverting a renaming quadruple

We have defined a renaming function, which works differently on two different cases. However, these two cases have in common that they behave like the identity function after a certain threshold. We use this property, combined with the first case of the rename function, to invert a renaming quadruple:

Definition 8.13 *Let* $t := (\overrightarrow{old}, \overrightarrow{new}, \overrightarrow{avoid}, \overrightarrow{domain})$ *be a well-formed renaming quadruple. Let* $M \in \mathbb{N}$ *be such that*

$$\forall x \geq M. rename(t, x) = x$$

Let $\vec{l} := [0..M-1]$ *be the sorted list containing all numbers from* 0 *to* $M-1$*, thus* $length(\vec{l}) = M$*. Let* $\vec{l'}$ *be the list such that*

1. $length(\vec{l'}) = M$
2. $\forall x < M. \vec{l'}[x] = rename(t, x)$

Let $\vec{d}$ *be any list of natural numbers (we just need to choose one). Then*

$$renameInv(t) := (\vec{l'}, \vec{l}, [], \vec{d})$$

Since we have $length(\vec{l}) = length(\vec{l'})$, and

$$set(\vec{l}) = set(\vec{l'}) = \{x | x \in \mathbb{N} \wedge x < M\}$$

by definition, our inverted renaming quadruple always falls into the first case of the renaming function (since *avoid* is empty):

$$\forall x \in \mathbb{N}. rename(renameInv(t), x) = rename_1(renameInv(t), x)$$

Using this property, we can show that our *renameInv* function satisfies the required properties:

Lemma 8.14 *Let* t *be a well-formed renaming. Then* $renameInv(t)$ *is also well-formed, and*

$$\forall x \in \mathbb{N}. rename(renameInv(t), rename(t, x)) = x$$

8.4.3 Towards the rename interface

Lifting the previous results to build a rename interface is quite obvious from this point, we therefore do not go into details. We need two functions, *toString* and *toNatural*, which respectively convert a natural number to a string and a string to a natural number. These functions are bijections, and in particular they are inverse of each other. Using these two functions, we can

convert any string renaming quadruple into a number renaming quadruple and vice versa. It is straightforward to instantiate the *rename* and *renameInv* functions of the interface. Renaming a state is simple. It suffices to use this function to modify the store s in the following way:

$$renameState(t, (s, \pi, h)) := (s', \pi, h)$$

where $domain(s') = \{rename(t, x) | x \in domain(s)\}$ and

$$\forall x \in domain(s).s'(rename(t, x)) := s(x)$$

The instantiation of *renameCustom* is also straightforward given a custom type of statements.

8.5 Custom statements

The purpose of the custom interface is to encode all statements which are not built in the parametric language. As an example, the parametric language defines neither **`assert`** statements nor variable assignments. Moreover, it only defines a special case of **`inhale`** and **`exhale`**. This section illustrates how to use the custom interface. We first show how to define **`assert`** and variable assignment, and then discuss how to encode **`inhale`** and **`exhale`** in the general case.

We assume we have a function (not described here) $\lambda e.wdef(e, \varphi)$ which is true if and only if e (an expression or a syntactic assertion) is well-defined in the state φ. We also assume we have a function $\lambda e.eval(e, \varphi)$ which evaluates a boolean expression or a syntactic assertion in a certain state. Since the store is a mapping, it is easy to encode a variable assignment using these two functions:

Definition 8.15 ***Variable assignment***

$$semanticsCustom_{Pr}((s, \pi, h), x := e)$$
$$:= \begin{cases} \{(s[x := eval(e, (s, \pi, h))], \pi, h)\} & \text{if } wdef(e, (s, \pi, h)) \wedge x \in domain(s) \\ Error & \text{otherwise} \end{cases}$$

A variable assignment $x := e$ verifies if and only if an expression is well-defined in the state (s, π, h) and x is already defined by s. If it verifies, then it simply modifies the mapping of x, by evaluating the expression in the state (s, π, h).

To define **`assert`**, **`inhale`** and **`exhale`**, we need to embed the type of syntactic assertions into the type O of custom statements. We can then encode the semantics of **`assert`** as follows:

Definition 8.16 *Assert*

$$semanticsCustom_{Pr}(\varphi, \textbf{assert } P) := \begin{cases} \{\varphi\} & \text{if } wdef(P,\varphi) \wedge eval(P,\varphi) \\ Error & \text{otherwise} \end{cases}$$

assert verifies if and only if the assertion is well-defined and true in the state φ. If it verifies, it does not modify the state.

We do not give a description for the semantics of **`inhale`** and **`exhale`**, since they involve some subtleties which are beyond the scope of this thesis. We refer the reader to [6] for a first approach on how to define these semantics, as well as [20] for a more theoretical approach. The important message is that these statements can be encoded using syntactic assertions (embedded in the type O) and custom statements, and they behave in general differently than the **inhale** and **exhale** statements provided by the parametric language. As an example, **`inhale acc(x.f, perm(x.f))`** encoded with the **inhale** statement from the parametric language and with a semantic assertion would simply be equivalent to **skip**, whereas in Viper it would double the permission held to $x.f$.

8.6 Completeness: Strongest postcondition in Viper

To leverage the completeness result of Chapter 7, we need to prove the strongest postcondition assumption for Viper. That is, we need to show how to construct an assertion which completely characterizes a Viper statement, and then to prove it. As we saw with the "permission gap" example, this assumption is false in general. We therefore sketch how to prove this assumption, assuming that we can use existential permissions in Viper.[1] We sketch in this section a proof for the strongest postcondition assumption in Viper.

The idea is to prove the assumption by induction on a statement s, with a strongest postcondition of the following form:

$$\exists \vec{v}.behavior(\vec{v}) \wedge \tag{8.1}$$
$$x = v_0 \wedge y = v_7 \wedge z = v_9 \wedge \ldots \wedge \tag{8.2}$$
$$acc(x.f, v_2) \wedge (v_2 > 0 \Rightarrow x.f = v_5) \wedge \ldots \tag{8.3}$$

8.1 fully specifies the behavior of the program (assumptions, conditional branching, permissions inhaled, values assigned...), in terms of existentially quantified variables. 8.2 links the existentially quantified variables $\vec{v}$ to the

[1] This can easily be done for permissions to field locations, but extending this idea to predicates and magic wands seems more complicated.

Listing 8.1: Example of a program for which we construct a strongest postcondition.

```
field f: Int

method example_strongest_post()
{
    var m: Int
    var x: Ref
    var n: Int

    assume n >= 2
    if (m >= 5) {
        n := 2 * n
    }
    inhale acc(x.f, 1/n)
    n := 0
}
```

$$
\begin{aligned}
&\exists m_0 : Int, x_0 : Ref, p_0 : Perm, p_1 : Perm, xf_0 : Int, && (8.4)\\
&\quad n_0 : Int, n_1 : Int, n_2 : Int, && (8.5)\\
&\quad e_0 : Bool, e_1 : Int, e_2 : Perm, e_3 : Int. && (8.6)\\
&\quad p_0 = 0 \wedge n_0 \geq 2 \wedge && (8.7)\\
&\quad (e_0 \Leftrightarrow m_0 \geq 5) \wedge e_1 = 2 * n_0 \wedge && (8.8)\\
&\quad (e_0 \Rightarrow n_1 = e_1) \wedge (\neg e_0 \Rightarrow n_1 = n_0) \wedge && (8.9)\\
&\quad e_2 = 1/n_1 \wedge p_1 = p_0 + e_2 \wedge e_3 = 0 \wedge n_2 = e_3 \wedge && (8.10)\\
&\quad m = m_0 \wedge x = x_0 \wedge acc(x.f, p_1) \wedge && (8.11)\\
&\quad (p_1 > 0 \Rightarrow x.f = xf_0) \wedge n = n_2 && (8.12)
\end{aligned}
$$

Figure 8.1: Constructing a strongest postcondition in Viper.

variables of the program (x, y, z, ...). Finally, 8.3 links the existentially quantified variables to permissions held to field locations and the values of those fields (if the permission held is greater than 0). As discussed in Remark 7.2, an assertion of this shape, using existential permissions, would only satisfy the strongest postcondition assumption under the soundness condition.

To prove the strongest postcondition assumption by induction with this form, we add information to 8.1, by adding new existential variables and new conjunctions. We then modify 8.2 and 8.3 accordingly.

We show an example of a strongest postcondition we could construct for Listing 8.1 in Figure 8.1. We have existential variables for all variables of

the program, with as many versions as the number of times this variable could potentially be modified. For example, n has an initial value, and can be modified on lines 11 and 14, hence three versions n_0, n_1, and n_2. We also have existential variables for evaluating expressions: e_0 is the condition of the **if** statement, e_1 represents $2 * n$ of line 11, e_2 is the $1/n$ of line 13, and e_3 is 0 of line 14.

All lines except 8.11 and 8.12 specify the behavior of the program. 8.7 specifies that we have initially no permission to $x.f$, and encodes the **assume** statement (line 9), 8.8 and 8.9 encode the conditional branching with its assignment (lines 10 to 12). Finally, 8.10 encodes the last two lines of the method, the **inhale** statement and `n := 0`.

As we can see from this example, the size of this postcondition can grow really quickly. This is not an issue since we are only interested in a theoretical construction which allows us to show completeness. Other difficulties arise when trying to prove this assumption, which are not shown in this example, but which can be dealt with, amongst which:

- **Aliasing**: When assigning a value to $x.f$, we should encode that this can also modify the value of $y.f$, $z.f$, ...
- **Havocing when no more permission**: When the permission held to a location goes to 0, we need to forget the value of this location.

Viper magic wands and predicates (which we do not consider in the simplified subset defined in Section 8.1) also present another challenge, since they can hide information which can be lost when they are transferred through annotations. Consider Figure 8.2 as an example for predicates. We have a predicate $P(x)$, which contains full permission to $x.f$. Listing 8.2 shows the original annotated program, which does not verify because line 20 fails. We know that $x.f = 5$ because of line 10. This information is then hidden in the predicate $P(x)$, line 12. However, since we put this predicate in the loop invariant (line 14), we get back *a* predicate $P(x)$, but we are not sure it is the same predicate as the one we gave to the loop through the loop invariant. Therefore, we lost the information that $x.f = 5$.

On the other hand, the inlined program (bound of 1) shown on Listing 8.3 verifies. Indeed, we are sure that we have the same version of the predicate $P(x)$ throughout the method, so we know $x.f = 5$ after having unfolded the predicate. One way to solve this issue is to uncomment the loop invariant on line 15. This loop invariant make explicit the hidden information $x.f = 5$ from $P(x)$. We also would have to do something similar for magic wands.

Moreover, even if recursive predicates can be stactically unbounded, we know that the number of times any predicate is unfolded in the bounded program is stactically bounded. Therefore, we only need to encode information into annotations up to a bounded depth, by using nested **unfolding**.

Listing 8.2: Original annotated program.

```
field f: Int

predicate P(x: Ref) {
    acc(x.f)
}

method example(x: Ref)
    requires acc(x.f)
{
    x.f := 5
    var i: Int := 0
    fold P(x)
    while (i < 1)
        invariant P(x)
        // && unfolding P(x) in x.f == 5
    {
        i := i + 1
    }
    unfold P(x)
    assert x.f == 5
}
```

Listing 8.3: Inlined (bound of 1).

```
field f: Int

predicate P(x: Ref) {
    acc(x.f)
}

method example(x: Ref)
    requires acc(x.f)
{
    x.f := 5
    var i: Int := 0
    fold P(x)
    if (i < 1)
    {
        i := i + 1
        assume !(i < 1)
    }
    unfold P(x)
    assert x.f == 5
}
```

Figure 8.2: Viper predicates can hide information.

To conclude, it seems possible to prove this strongest postcondition assumption in Viper. However, we need to extend assertions with existential permissions. Such a proof would be a complicated one and needs more exploration.[2]

8.7 Towards two useful features in Viper

8.7.1 Static inlining for early error detection

As explained in the introduction, a major motivation for this project is the creation of a useful feature for early error detection for Viper. This feature would not only detect fundamental errors, but could also provide more information on spurious errors (as we explain in Chapter 9).

We can leverage some theoretical results obtained in this thesis to detect fundamental errors in Viper. First, we need to ensure that the soundness condition is satisfied. This mostly requires a way of checking whether a statement is **mono** or **framing**.

[2] Another difficulty is dealing with Viper functions, which represent abstraction over expressions.

One interesting result is that these properties are stable by sequential composition and non-deterministic conditional branching. Moreover, Viper statements which are not **mono** (and thus not **framing**) necessarily use **perm**, **forperm** or contain an **assume** statement with an accessibility predicate. Viper statements which are not **framing** necessarily satisfy the latter condition, or contain an **exhale** statement with a **wildcard**.[3] Combining these two results gives a straightforward conservative function to check the soundness condition. Leveraging the definition of inlining and the soundness result, we can report fundamental errors.

8.7.2 Speed up re-verification through caching

This work yields an unexpected byproduct, which relates to caching verification results. Thanks to the modular verification of method calls, modifying the body of a method without modifying its annotation does not require one to re-verify all calls to this method. However, in Viper, it is unclear what we should do when a precondition or a postcondition is modified.

Thanks to the properties **mono** and **framing**, we can explore and express answers to this. We give some examples here:

- If we strengthen the precondition of a method, and if this method's body is **mono**, then we do not need to check again whether the method satisfies its contract.
- If we weaken the precondition of a method and the method still satisfies its contract, then we do not have to check again the program containing the method call if the statement after the method call is **mono**: Indeed, a weaker precondition with the same postcondition gives a stronger set of states, which still verifies in this case.
- Conjoining (with the separation conjunction) the same assertion to both the precondition and the postcondition of a method: We do not have to check the method again if its body is **framing**.

This problem will be explored deeper in a future work.

[3] As an example, **exhale acc**(x.f, **wildcard**) is **mono** and not **framing**, but **inhale acc**(x.f, **wildcard**) is **framing** (and thus **mono**). See [1] for more details about **wildcard**.

Chapter 9

Extensions to Different Loop Semantics and Inlinings

This chapter presents two possible extensions to the results already presented (soundness and completeness), based on Viper. The first section explores three different possibilities for defining the semantics of a loop in Viper. The second section explores three useful ways of inlining, corresponding to different stages of annotating a program.

9.1 Exploration of loop semantics

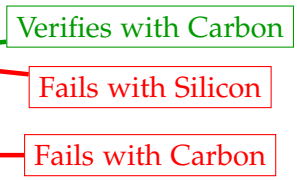
Our soundness and completeness results only apply to one possible semantics for loops, as expressed in Definition 5.33. We refer to this semantics of loops as black-box semantics, since the verification of loops in this semantics does not take into account impure resources from outside the loop, except the ones explicitly specified in the loop invariant. For example, if full permission is held to both $x.f$ and $y.f$ before the loop, and the loop invariant only specifies **acc**(x.f), then the loop body completely ignores the full permission to $y.f$, and behaves as if this permission never existed.

However, this black-box semantics is only one of several possibilities to define loop semantics, and our general framework can only deal with this one. In this section, we explore two other possibilities for defining the semantics of loops in Viper. Based on this exploration, it should be possible to also define these different loop semantics in our parametric language, and prove the soundness of inlining for all of them.

This section only focuses on the exploration of different loop semantics in Viper. We first highlight a difference between how Silicon and Carbon, two verifiers for Viper, treat loops. Based on this difference, we present two more possibilities for loop semantics, on top of the black-box semantics. We only present these three different loop semantics, but no result of soundness or

Listing 9.1: Differences of loop treatment between the Viper verifiers Carbon and Silicon.

```
1  field f: Int
2
3  method example(x: Ref, y: Ref, n: Int)
4          requires acc(x.f, 1/2) && x.f == 5
5  {
6      var i: Int := 0
7      while (i < n)
8          invariant true
9      {
10         inhale acc(x.f, 1/3)
11         assert x.f == 5
12         exhale acc(x.f, 1/3)
13         inhale acc(x.f, 1/3)
14         assert x.f == 5
15         exhale acc(x.f, 1/3)
16         i := i + 1
17     }
18 }
```



completeness of inlining have been formalized yet with respect to the two new possibilities.

9.1.1 Semantics of loops in Viper: Silicon and Carbon

Listing 9.1 presents a Viper program where the two Viper verification back-ends, Silicon and Carbon, differ in their treatment of the loop. Before entering the loop, the program holds half permission to $x.f$, and the value of $x.f$ is 5.

The Silicon back-end actually treats loops as black-box, and thus line 9 does not verify with Silicon. Indeed, since the loop invariant given here (**true**) does not give any information, Silicon does not know anything about the value of $x.f$ inside the loop.

On the other hand, line 9 verifies with the Carbon back-end. It is justified by the fact that the program holds half permission to $x.f$, and this permission has not been transferred to the loop through the loop invariant. Therefore, this permission is still held by the program state after the loop, and so does $x.f == 5$. The information $x.f == 5$ is true before and after the loop. The reason is that the Carbon back-end assumes that this heap value cannot be modified from inside the loop body, since half permission to this heap location is kept outside the loop. Following this assumption that this value cannot be modified from inside the loop body, Carbon considers this information true inside the loop body.

However, line 12 suprisingly does not verify with Carbon. The practical

reason is that the **exhale** line 12 removes all permission held (inside the loop body) to $x.f$, and thus the value of $x.f$ is havoced. It would however be coherent for line 12 to verify, using the same reasoning as before. This is the idea developed in the second possible loop semantics.

9.1.2 Three coherent ways to treat loops

We now present three coherent ways to define the semantics of loops in Viper. The first is the black-box semantics, the one we used in our framework, and the one used by the Silicon back-end. The second one is the logical extension of how Carbon treats loops. In Carbon, impure resources not specified in the loop invariant, which are accessible after the loop, are also accessible from inside the loop. Finally, the last loop semantics builds on top of the second one, and argues that giving access to reading permissions from outside the loop is coherent with respect to the role of a loop invariant.

Black-box semantics

The black-box semantics for loops is the one from Definition 5.33 and described in Section 2.2.4 of the background chapter. In this semantics, the loop body has only access to the values of variables which were defined before the loop, and the impure resources specified by the loop invariant. It has no access to the impure resources, namely the permissions to and the values of heap locations, which are not specified in the loop invariant.

Access to knowledge of the heap

Viper semantics can be seen as based on a global assumption about the program: At any point of the execution, the sum of all permissions held to a field location in a program sums to at most 1. A corollary is that, if some permission to a field location is kept outside a loop, then the knowledge about this field can be framed around the loop. The method shown on Listing 9.1, for example, keeps half permission to $x.f$ outside the loop. We therefore know that, after the loop, $x.f$ still equals 5.

Logically extending this global assumption, we could allow the body of a loop to have access to impure resources not specified in the loop invariant. Carbon already does this, but only to a certain degree, as we illustrated on Listing 9.1. In practice, lines 9 and 12 on Listing 9.1 would both verify with this semantics for loops.

This also implies that the permissions held to a field location, inside and outside a loop, should sum to at most 1. Take Listing 9.2 as an example to illustrate this idea. Line 14 does not verify with Carbon. Half permission to $x.f$ is kept outside the loop, and half permission is given to the loop. Line 12 verifies, since $x.f$ is known to be 5 outside the loop, with some permission.

Listing 9.2: Example with too much permission inside and outside a loop.

```
1  field f: Int
2
3  method example(x: Ref, n: Int)
4      requires acc(x.f) && x.f == 5
5  {
6      var i: Int := 0
7      while (i < n)
8          invariant acc(x.f, 1/2)
9          // invariant i >= 1 ==> x.f == 4
10     {
11         inhale acc(x.f, 1/2)
12         assert x.f == 5
13         x.f := 4
14         assert i >= 1 ==> false
15         i := i + 1
16     }
17 }
```

However, at every loop iteration, we inhale enough permission to $x.f$ to write it, and we assign 4 to it. Therefore, from the second iteration on ($i \geq 1$), we should know that $x.f$ is both 4 and 5, which is incoherent and thus would make line 14 verify. We can make this program verify by uncommenting the loop invariant on line 9, showing that this behavior is, in a way, already implied by how Carbon treats loops.

Role of a loop invariant and access to read permissions outside the loop

We can take this idea of making what is framed around the loop accessible from inside the loop body even further. Since we frame permissions around the loop, why not also make these permissions accessible from inside the loop body? As an example of what this would mean, take Listing 9.3. To make this program verify, we need in particular to make line 11 verify. In the black-box semantics, we need to uncomment both invariants on line 8 and 9. In the previous semantics (the one inspired by Carbon), we only need to uncomment the invariant on line 8. If we would allow access to read permissions outside the loop, then this program would verify without any invariant.

This raises the question of the role of a loop invariant. As explained in the introduction (Chapter 1), we have to annotate loops with loop invariants to make verification decidable. The role of a loop invariant in general is therefore to approximate the behavior of a loop.

In Viper, a loop invariant is not exactly an approximation of the behavior of a loop. A loop invariant in Viper also defines a transfer of resources,

Listing 9.3: Example of a program which would verify in the third loop semantics.

```
field f: Int

method example(x: Ref, n: Int)
    requires acc(x.f) && x.f == 5
{
    var i: Int := 0
    while (i < n)
        // invariant acc(x.f, 1/2)
        // invariant x.f == 5
    {
        assert x.f == 5
        i := i + 1
    }
}
```

Listing 9.4: Example of a loop whose behavior depends on the loop invariant.

```
while (i < n)
    invariant ...
{
    p := perm(x.f)
    i := i + 1
}
```

and especially impure resources. As an example, take the loop shown on Listing 9.4. The behavior of this loop body depends on the loop invariant. Indeed, it assigns to p the permission held to $x.f$ inside the loop, and this permission depends on the loop invariant. If the loop invariant is **true**, then $p = 0$, whereas $p = 1/2$ if the loop invariant is **acc**(x.f, 1/2).

One could think that the loop invariant should not influence the behavior of the loop. In this case, all permissions from outside the loop should be accessible from within the loop. However, a loop invariant should still be allowed to use accessibility predicates (**acc**(...)), for two reasons:

1. If some permission inside the loop is inhaled or exhaled, then the permission mask is modified. This modification is part of the program, and it should be possible to approximate it using the loop invariant.

2. Viper assumes that if some permission to a heap location is kept outside the loop, its value is not modified inside the loop. This form of framing would be lost if all permissions held outside the loop are accessible from inside the loop. Thus, permissions held outside the loop should only allow reading a heap location, but not writing it. Full permission

inside the loop should still be required to write a heap location.

Therefore, only allowing to read a field location, when some permission to this field location is detained, inside or outside the loop, does not have this issue. Moreover, as shown on Listing 9.3, it would reduce the size of loop invariants. This is another kind of loop semantics to explore.

9.2 Different inlinings: A classification of inlinings in Viper

In this section, we show that there is not only one useful way to inline a program, but many. In particular, we first focus on Viper programs to illustrate how three different scopings for inlining could be useful, at different stages of annotating the program. These three scopings are related to the three loop semantics presented in the previous section. We present in this section a general way of describing and classifying inlinings, based on two concepts: barriers and scopings.

9.2.1 A chronology of annotating

In this subsection, we give a simple description of annotating a program in Viper, based on experience, in three stages:

First stage: Writing the program. This is the situation we described in the introduction. We have a program, without any annotation, and we want to know if it is possible to find an annotation such that it verifies.

Second stage: Annotating the program with permissions. Once the program is written, Viper reports a lot of errors because of insufficient permissions. Since no permission is specified in preconditions, postconditions and loop invariants, no permission is transferred to loops and methods. This stage consists of writing all the required permissions to describe all transfers of permission to loops and methods, to remove the spurious errors reported by Viper.

Third stage: The interesting properties. Once the program verifies with respect to permissions, one can begin to write interesting properties, which describe the behavior of the program. This stage consists mostly in adding annotations describing values of variables and fields.

Partly based on this chronology, we define in the following subsections three different scopings for inlining, which correspond to these three stages. Using these different scopings, we could define an inlining function which takes annotations into account, and provides more information about an error to

Listing 9.5: A method call.

```
method initial() {
    ...

    m()

    ...
}
```

Listing 9.6: An inlined method call.

```
method initial() {
    ...  Barrier 1: precondition
    body_m
         Barrier 2: postcondition
    ...
}
```

≈ framing

Figure 9.1: Illustration of the concept of barriers.

the programmer. We first need to define the concept of a barrier. This section is based on the black-box semantics of loops in Viper.

9.2.2 Barrier

We illustrate the concept of a *barrier* in Figure 9.1. A barrier acts like a loop invariant at the beginning of a loop: The loop invariant takes some permissions, specified by the annotation, transfer these permissions to the loop body, and let the remaining permissions outside the loop. Similarly, a barrier is parametrized by an annotation, which characterizes a transfer of permissions and information. Listing 9.5 shows a method call, and Listing 9.6 the inlined version of this method call, where we represent two barriers as blue lines. The first barrier is parametrized by the precondition of the method, while the second barrier is parametrized by the postcondition. These two barriers are connected, which also defines what is framed. An inlining simply defines what these barriers filter, what gets through and what is framed, with respect to the annotation they are parametrized with.

The inlining function from Definition 6.3 simply lets everything go through. It acts as if using barriers, where the annotations capture everything about the program state. Since both barriers, in this case, behave in the same way with different annotations, we refer to their scopings as *no scoping* (defined below). However, it is possible to define other scopings for these barriers.

An example of another scoping is to only let the permissions specified by the annotation go through the barrier. The permissions not specified by the annotation are either framed (in the case of barrier 1), or leaked (in the case of barrier 2). The next subsection defines more precisely such scopings.

This concept of barriers can be extended to loops, with a barrier at the entrance of the loop, a barrier at the exit of the loop, and intermediate barriers in between every iteration.

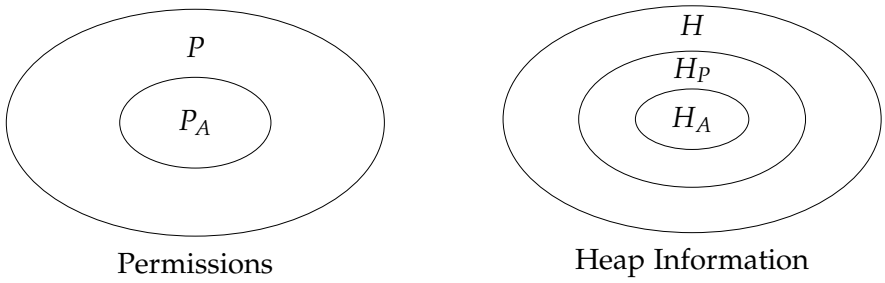


Figure 9.2: Degrees of freedom for scoping of barriers

9.2.3 Scoping of a barrier

The scoping of a barrier defines what gets through and what does not, based on its annotation. What does not get through is either framed (precondition barrier for a method, entrance barrier for a loop) or leaked (postcondition for a method, intermediate and exit barriers for a loop).

A Viper state consists of a store, a permission mask and a heap (see Section 2.2.2). Variables, permissions and heap values are the three types of information we can filter through a barrier. Since variables are accessible from within the body of a loop, we do not filter out any information about variables. We are therefore left with permissions and heap information.

We define two levels of permission scoping, and three levels of heap information scoping, as shown in Figure 9.2. P represents the whole permission mask, whereas P_A represents the part of the permission mask described by the annotation A. H represents the whole heap, H_P the part of the heap for which permission is held from A, and H_A represents the heap information exactly defined by A.

Consider, as an example, the assertion $A :=$ `acc(x.f, 1/2) * x.f >= 3`, and the state φ with full permission to both x and y, and where $x.f == 5$ and $y.f == 7$. We have, in this settings:

- $P = acc(x.f) * acc(y.f)$, that is full permission to both $x.f$ and $y.f$.
- $P_A = acc(x.f, 1/2)$, that is half permission to $x.f$.
- H represents the full heap, where $x.f == 5$ and $x.f == 7$.
- H_P represents the information $x.f = 5$, but does not contain any information about the value of $y.f$.
- H_A represents represents the information $x.f \geq 3$.

Using this classification of information, we define three different scopings, corresponding to the three stages of annotating presented earlier.

No scoping: H and P. This is the scoping used in Definition 6.3. Since H is the full heap, and P the full permission mask, it lets everything go through, and therefore does not frame anything. As we saw, inlining with no scoping barriers is useful when no annotations have been written yet.

Weak Scoping: H_P and P_A. The second stage of annotating corresponds to a point where annotations define the transfers of permission. In this case, it is not useful anymore to inline by letting the full permission mask go through a barrier. This scoping therefore lets P_A go through. However, since the annotation does not contain any heap information yet, we give all of the heap information: H_P. Our hope is that we can prove a strong soundness result for this scoping, which does not require a soundness condition. Indeed, this soundness result would not speak about annotating the program, but about adding information to the already existing annotation, in the form of store and heap information. It also seems that we can prove a stronger completeness result, where we do not need the soundness condition, and we do not need to bridge the permission gap.

Strong Scoping: H_A and P_A. The last stage of annotating corresponds to adding interesting properties, in particular heap information. When all the heap information needed has been specified in the annotation, the only remaining thing one can add is information about the store. It seems, similarly to *weak scoping*, that we can prove strong soundness and completeness results, speaking about adding only information about the store to annotations.

These three scopings can also be combined to get hybrid forms of inlining. When, for example, one has finished writing a precondition for a method, but no postcondition has been written, we can inline with a strong scoping (or a weak scoping) barrier for the precondition (i.e., taking the precondition into account), and a no scoping barrier for the postcondition (i.e., letting all the permissions and information flow through the barrier).

Moreover, even when no annotation has been written, it is useful to use different scopings for inlining. This can give us more information on an error. We could, for example, report errors with messages such as:

- "Line 10 does not verify because of a permission error, it seems however that there is enough permission in the surrounding scope" (if no scoping verifies, but weak scoping does not).
- "There seems to be insufficient permission in the surrounding scope for line 10 to verify, in the third iteration of the loop" (if no scoping does not verify, with the soundness condition).
- "The assertion on line 10 seems true, but heap information needs to be added to the precondition" (if weak scoping verifies, but strong scoping

does not).

These different combinations should be explored in the future, as well as the soundness and completeness of these different scopings, to build a useful feature in Viper.

Chapter 10

Conclusion and Future Work

10.1 Conclusion

We have explored static inlining in Viper, defined a soundness property with respect to annotations and fundamental errors, and shown examples of Viper programs where this soundness property does not hold. We have derived from these examples a *soundness condition*, namely a sufficient condition under which this soundness property holds, based on two properties of statements, **mono** and **framing**, as illustrated in Figure 10.1.

We have defined a general framework to abstract Viper, consisting of a

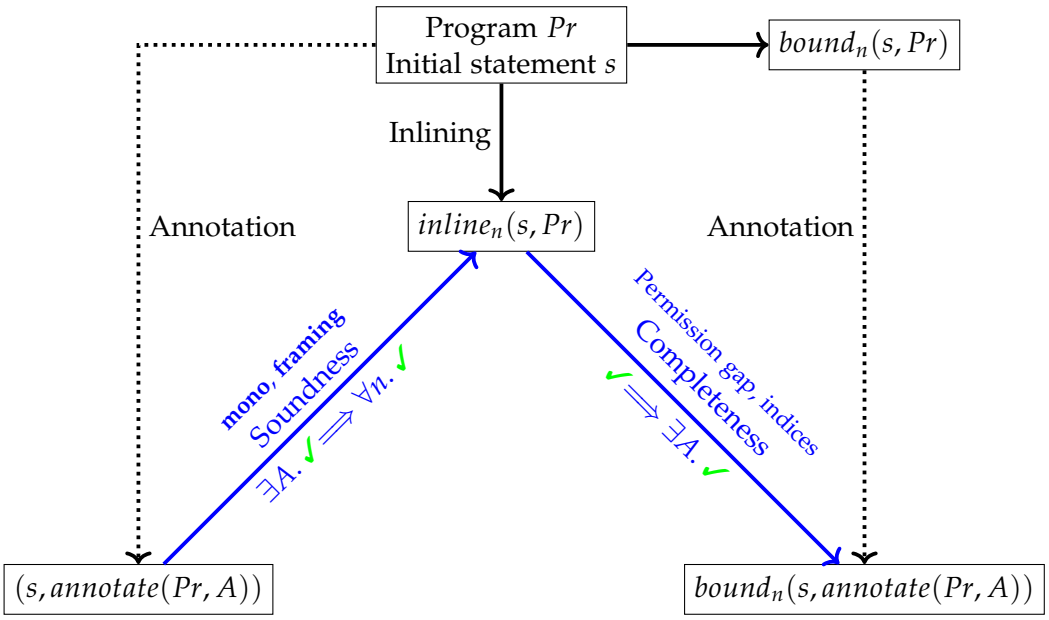


Figure 10.1: Illustration of the soundness and completeness results.

separation algebra and a parametric language. This separation algebra builds on other works on separation algebra, with a difference: We combine pure and impure resources in a single algebra.[1] This enabled us to define a parametric language centered around the semantics of inhaling and exhaling annotations, mimicking both transmission of pure information and transfer of impure resources.

This parametric language has many degrees of freedom. It can be instantiated as a model of a concrete verification language, given the right input parameters. We have formally defined inlining up to a bound, the *soundness condition* and the soundness property for this framework. We have formally proved the soundness of static inlining in this framework, and this proof has been mechanized with the proof assistant Isabelle/HOL (Appendix A).

We have also defined and explored completeness of inlining. We have shown examples of Viper programs where completeness does not hold, and identified two main issues, the lack of indices in loops and method calls, and a permission gap. We have also sketched a proof for completeness of inlining in our general framework, based on an assumption of a strongest postcondition.

To show how (and that) our general framework can be instantiated, we have discussed how to instantiate it for Viper. We have shown an example of how to instantiate the rename interface (proved in Isabelle/HOL), how to instantiate the other interfaces to get a model of the Viper language, and how this instantiation can be leveraged to build two useful features in Viper: an early error detection feature based on static inlining, and a cache re-verification feature based on **mono** and **framing**.

Finally, we have proposed and discussed two extensions for this work. We have explored three different yet coherent possible definitions of loop semantics. Our work deals with one kind of loop semantics, and could be extended to the other two. Moreover, we have shown that different inlinings are possible, that they correspond to different stages of annotating a program, and that combining them can yield a lot of useful information in the early stages of annotating.

10.2 Future work

As we explained throughout this thesis, this work has opened many interesting directions, some of which should be addressed in our future work.

[1] As noted in the related work section of Chapter 4, Iris also does this.

10.2.1 Improve the current framework and soundness proof

Loops semantics. As explained in Remark 5.34, the definition of loop semantics in this thesis is slightly different than the one formalized in Isabelle/HOL. The semantics of **while** (b) **inv** I $\{s\}$ for a state φ should havoc $modif(s) \cap \sigma(\varphi)$, but the Isabelle formalization havocs only $modif(s)$, which makes it impossible to declare variables in a loop body. This should be fairly easy to fix.

Variable capturing when inlining As explained in Remark 6.4, the current definition of the inlining function in the case of a sequential composition allows variable capturing. The second recursive call should be modified to disable variable capturing.

Soundness condition. As explained in Remark 6.12, Definition 6.11 defines $SC_0^{Pr}(l, \textbf{while } (b) \textbf{ inv } I \ \{s\})$ as $mono_{Pr}(\textbf{assume } \neg b)$. However, the Isabelle/HOL formalization defines it as $wfm_{Pr}(b) \wedge mono_{Pr}(s)$, because a first version of the proof of soundness required s to be **mono** under the soundness condition. The proofs presented in Chapter 6 show that this is no longer needed, and these proofs should replace the ones in our formalization.

Well-formed assertions The current definition of **well-formed** assertions is too restrictive and disallows some Viper annotations. This definition should be modified as explained in Remark 5.21.

Extended semantic assertions. As explained in Section 5.3.3, there is a mismatch between syntactic and semantic assertions, and our current model is therefore limited. A solution to tackle this issue is to use extended semantic assertions, which can return an *error* on top of *true* and *false*, to model when assertions are not well-defined. This solution should be explored, as well as its relationship with annotations in Viper. This solution should then be used to solve this mismatch.

10.2.2 Completeness of inlining

Completeness in our general framework. Chapter 7 only sketches a way of how to construct annotations for proving completeness. Completeness in this framework under the strongest postcondition assumption should be explored deeper. In particular:

1. The notion of bounded program should be formalized, maybe as a semantic notion instead of a syntactic one.
2. The completeness property should be formally defined, as well as what it means to annotate a program.

3. The gap between completeness with indices for all methods and the examples where completeness does not hold when recursive methods are not indexed should be bridged. That is, we should either find an example of incompleteness because of a non-recursive method which is not indexed, or we should prove completeness even in the case of non-recursive methods which are not indexed.
4. A completeness theorem, similar to the soundness theorem, should be formalized and proved in Isabelle/HOL, using the same framework.

Completeness in Viper. Section 8.6 sketches a proof for the strongest postcondition assumption in Viper.

1. The idea of existential permissions in Viper should be formalized, given a semantics, and the supportedness property for a semantic assertion in our general framework should be adapted consequently. Otherwise, another approach should be developed to bridge the permission gap, for example, approaches using abstract read permissions [9, 22] or ghost variables.
2. The strongest postcondition assumption should be formalized and proved, at least for a simplified version of Viper.

10.2.3 Towards useful features in Viper

Instantiation of the general framework with Viper. The injection of Viper described in Chapter 8 is not complete.

1. The semantics of Viper should be formally defined.
2. The instantiation should be done with a more complex version of Viper, taking into account for example trace semantics, magic wands, and predicates.
3. The requirements for the instantiation with Viper should be proved.

Early error detection in Viper. The results proved or sketched in this thesis should be leveraged in Viper.

1. A function to check whether a Viper statement is in **mono** or **framing** should be developed in a conservative way, using the stability of these properties with sequential composition and conditional branching. `perm`, `forperm` and `assume` statements should be explored further to make this feature more precise.
2. Based on the notion of a barrier and the different scopings described in Chapter 8, several functions for inlining in Viper should be developed.

3. These functions should be used for developing a feature to provide more useful information on errors reported in early stages of annotating a Viper program.
4. Extensions of this feature should be considered, such as stratified inlining [13, 12], adapting the inlining to the annotation (such as not inlining further when an annotation is already written, adapt the scoping based on the shape of the annotation...).

Caching verification results. As described in Section 8.7.2, this work opens the way for speeding up reverification[2] in Viper, using **mono** and **framing** to know when it is not necessary to verify some parts of the code again. This problem should be explored in depth.

Inlining in other contexts. This work focuses on static inlining at the Viper level. However, Viper serves as an intermediate verification language, on which many front-ends are built. It is unclear whether inlining at the front-end level is related to inlining at the Viper level. Inlining at the front-end level should be explored, as well as its relationship to inlining at the Viper level.

[2]That is, verification after small modifications of a first verified version.

List of Figures

Listings

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Appendix A

Appendix: Isabelle Formalization

A.1 Theories presented in this appendix

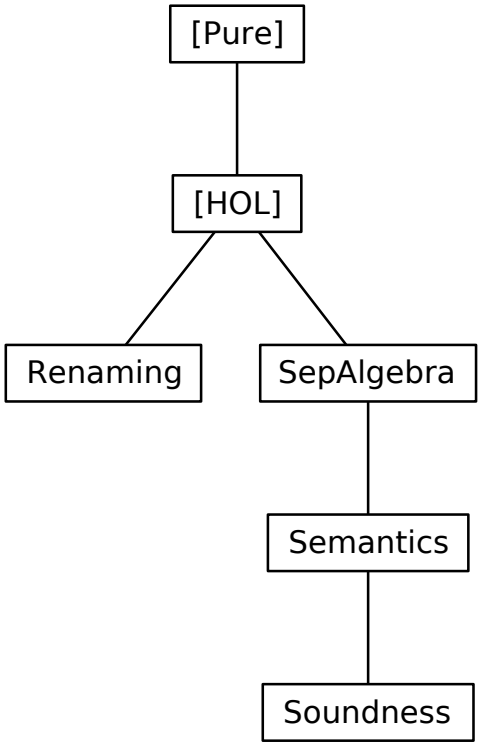

Figure A.1: Theories presented in this appendix

Figure A.1 shows the theories presented in this appendix. The *SepAlgebra* theory defines two locales, the preliminary separation algebra and the actual separation algebra, as defined in Chapter 4. It also defines and proves properties on semantic assertions. The *Semantics* theory defines the language and its semantics defined in Chapter 5, using a locale to express the parameters and the requirements which should be satisfied. The *Soundness* theory formalizes Chapter 6: It defines the static inlining function, the soundness condition, and formally proves the soundness theorem. Unrelated to the other theories, the *Renaming* theory formalizes the rename interface described in Section 8.4. For space reasons, we do not show the proofs, only the definitions and the lemmas.

A.2 Separation algebra

theory *SepAlgebra*
imports *Main*
begin

type_synonym *'a assertion = "'a ⇒ bool"*

A.2.1 Preliminary separation algebra

locale *pre_sep_algebra =*
fixes *orig_plus :: "'a ⇒ 'a ⇒ 'a option"*
fixes *u :: "'a"*
assumes *orig_comm: "orig_plus a b = orig_plus b a"*
and *asso1: "Some ab = orig_plus a b ∧ Some bc = orig_plus b c ⟹ orig_plus ab c = orig_plus a bc"*
and *asso2: "Some ab = orig_plus a b ∧ None = orig_plus b c ⟹ None = orig_plus ab c"*
and *asso3: "None = orig_plus a b ∧ Some bc = orig_plus b c ⟹ None = orig_plus a bc"*
and *orig_neutral: "orig_plus a u = Some a"*
begin

fun *plus :: "'a option ⇒ 'a option ⇒ 'a option"* (**infixl** *"⊕" 60*) **where**
"Some a ⊕ Some b = orig_plus a b"
| "_ ⊕ _ = None"

definition *defined :: "'a ⇒ 'a ⇒ bool"* (**infixl** *"##" 60*)
where *"defined a b ⟷ (¬ Option.is_none (Some a ⊕ Some b))"*

definition *smaller :: "'a ⇒ 'a ⇒ bool"* (**infixl** *"<<" 50*) **where**
"smaller a b ⟷ (∃ c. Some b = Some a ⊕ Some c)"

lemma *commutative:*
"a ⊕ b = b ⊕ a"
⟨*proof*⟩

lemma *associative:*
"(a ⊕ b) ⊕ c = a ⊕ (b ⊕ c)"
⟨*proof*⟩

lemma *neutral:*
"a ⊕ Some u = a" ⟨*proof*⟩

lemma *smaller_refl:*
"a << a"
⟨*proof*⟩

lemma *smaller_trans:*
assumes *"a << b"*

```
      and "b << c"
    shows "a << c"
⟨proof⟩

definition pure :: "'a ⇒ bool" where
  "pure φ ⟷ Some φ ⊕ Some φ = Some φ"

definition pure_set :: "'a ⇒ 'a set" where
  "pure_set φ = {φ'. pure φ' ∧ φ' << φ}"

definition core :: "'a ⇒ 'a" ("|_|")
  where "|φ| = the (Finite_Set.fold (⊕) (Some u) (Some ` (pure_set φ)))"

lemma pure_smaller_ok:
  assumes "p << φ"
      and "pure p"
    shows "Some φ = Some φ ⊕ Some p"
  ⟨proof⟩

lemma pure_add:
  assumes "pure a"
  assumes "pure b"
  assumes "a ## b"
  shows "pure (the (Some a ⊕ Some b))"
⟨proof⟩

lemma pure_set_stable_add:
  assumes "a ∈ pure_set φ"
      and "b ∈ pure_set φ"
      and "Some c = Some a ⊕ Some b"
    shows "c ∈ pure_set φ"
⟨proof⟩

lemma pure_u [simp]: "pure u" ⟨proof⟩

lemma u_smaller [simp]: "u << φ" ⟨proof⟩

lemma empty_in_pure:
  "u ∈ pure_set φ"
  ⟨proof⟩

definition s_core :: "'a ⇒ 'a option" where "s_core x = Some |x|"

end
```

A.2.2 Separation algebra

```
locale sep_algebra = pre_sep_algebra +
  fixes C :: "'a ⇒ 'a"
```

```
  assumes finiteness: "finite {φ'. pure φ' ∧ φ' << φ}"
      and partially_cancellative: "Some (C φ) ⊕ Some a = Some (C φ) ⊕
Some b ⟹ a = b"
      and decompo: "Some φ = Some |φ| ⊕ Some (C φ)"
      and c_empty_core: "|C φ| = u"
      and unique_c: "Some φ = Some |φ| ⊕ Some φ' ∧ |φ'| = u ⟹ φ' =
C φ"
      and positivity: "Some a ⊕ Some b = Some u ⟹ a = u"
      and pure_reducibility: "pure p ∧ p << a ∧ Some a = Some b ⊕ Some
c ⟹
          (∃a'. Some a' = s_core b ⊕ s_core c ∧ p << a')"
begin

lemma pure_u:
  "|u| = u"
  ⟨proof⟩

definition some_core :: "'a ⇒ 'a option" where
  "some_core φ = (Finite_Set.fold (⊕) (Some u) (Some ` (pure_set φ)))"

lemma commut_comp: "⋀y x. (⊕) y ∘ (⊕) x = (⊕) x ∘ (⊕) y"
  ⟨proof⟩

interpretation add_pure: comp_fun_commute "plus"
  ⟨proof⟩

lemma "fold_graph (⊕) (Some u) (Some ` pure_set φ) (some_core φ)"
⟨proof⟩

lemma "|φ| = the (some_core φ)" ⟨proof⟩

lemma pure_set_u: "pure_set u = {u}"
⟨proof⟩

lemma some_core_u:
  "some_core u = Some u"
⟨proof⟩

definition some_prop :: "'a ⇒ 'a option set ⇒ 'a option ⇒ bool" where
  "some_prop φ S s ⟷ (S ⊆ Some ` pure_set φ ⟶ (S = {} ∧ s = Some
u) ∨ (s ∈ Some ` pure_set φ ∧ (∀s' ∈ S. the s' << the s)))"

lemma some_prop_proof:
  "some_prop φ (Some ` pure_set φ) (some_core φ)"
⟨proof⟩

lemma core_is_max_uni:
  "|φ| ∈ pure_set φ ∧ (∀x ∈ pure_set φ. x << |φ| )"
⟨proof⟩
```

lemma *core_is_max:*
"φ' = |φ| ⟷ (φ' ∈ pure_set φ ∧ (∀φ'' ∈ pure_set φ. φ'' << φ'))"
(is *"?a ⟷ ?b")*
⟨*proof*⟩

lemma *sum_pure:*
assumes *"pure a"*
and *"pure b"*
and *"Some c = Some a ⊕ Some b"*
shows *"pure c"*
⟨*proof*⟩

lemma *add_trans:*
assumes *"a << aa"*
and *"b << bb"*
and *"Some cc = Some aa ⊕ Some bb"*
and *"Some c = Some a ⊕ Some b"*
shows *"c << cc"*
⟨*proof*⟩

lemma *core_add:*
assumes *"Some a = Some b ⊕ Some c"*
shows *"s_core a = s_core b ⊕ s_core c"*
⟨*proof*⟩

lemma *pure_set_finite:*
"finite (pure_set φ)"
⟨*proof*⟩

definition *s_C :: "'a ⇒ 'a option"* **where** *"s_C x = Some (C x)"*

lemma *core_properties:*
shows *"pure |φ|"*
and *"|φ| << φ"*
and *"pure φ ⟷ φ = |φ|"*
⟨*proof*⟩

lemma *not_pure_core:*
"¬ pure φ ⟷ φ ≠ |φ|" **(is** *"?a ⟷ ?b")*
⟨*proof*⟩

definition *add_set :: "'a set ⇒ 'a set ⇒ 'a set"* **(infixl** *"⊕⊕" 60)* **where**
"A ⊕⊕ B = {the (Some a ⊕ Some b) | a b. a ∈ A ∧ b ∈ B ∧ a ## b}"

lemma *elem_add_set:*
"x ∈ A ⊕⊕ B ⟷ (∃a b. a ∈ A ∧ b ∈ B ∧ Some x = Some a ⊕ Some b)"
⟨*proof*⟩

lemma *simple_set_add_comm:*
"A ⊕⊕ B = B ⊕⊕ A"
⟨*proof*⟩

lemma *core_inv:*
"||φ|| = |φ|"
⟨*proof*⟩

lemma *decompo_new_plus:*
"Some φ = s_core φ ⊕ s_C φ"
⟨*proof*⟩

lemma *properties_c:*
shows *"a = b ⟷ |a| = |b| ∧ C a = C b"*
and *"C (C φ) = C φ"*
⟨*proof*⟩

lemma *definedness [simp]:*
assumes *"a ## b"*
shows *"C a ## b"*
and *"core a ## b"*
and *"b ## a"*
⟨*proof*⟩

lemma *reduce_add:*
assumes *"a ## b"*
and *"Some a ⊕ Some b = Some a ⊕ Some c"*
shows *"s_core a ⊕ Some b = s_core a ⊕ Some c"*
⟨*proof*⟩

definition *intuitionistic :: "'a assertion ⇒ bool"* **where**
"intuitionistic P ⟷ (∀φ φ'. φ << φ' ⟶ (P φ ⟶ P φ'))"

definition *supported :: "'a assertion ⇒ bool"* **where**
"supported P ⟷ (∀φ. P φ ⟶ (∃m. m << φ ∧ P m ∧ (∀φ'. φ' << φ ∧ P φ' ⟶ (m << φ'))))"

definition *Inh :: "'a assertion ⇒ 'a set"* **where**
"Inh P = {φ. P φ ∧ (∀φ'. φ' << φ ∧ φ' ≠ φ ⟶ ¬ P φ')}"

definition *bigger_set :: "'a set ⇒ 'a set ⇒ bool"* (**infixl** *">>" 50*) **where**
"A >> B ⟷ (∀a ∈ A. ∃b ∈ B. b << a)"

lemma *rel_bigger_add_set:*
assumes *"A = B ⊕⊕ D"*
shows *"A >> B"*
⟨*proof*⟩

lemma *equiv_Inh [simp]:*

```
  assumes "P φ"
      and "⋀φ'. φ' << φ ∧ φ' ≠ φ ⟶ ¬ P φ'"
    shows "φ ∈ Inh P"
  ⟨proof⟩

lemma defined_trans_plus:
  assumes "Some a = Some b ⊕ Some c ⊕ Some d"
  shows "c ## d"
  ⟨proof⟩

lemma neutral_parts:
  shows "|u| = u"
    and "C u = u"
  ⟨proof⟩

lemma c_add:
  assumes "Some a = Some b ⊕ Some c"
  shows "s_C a = s_C b ⊕ s_C c"
⟨proof⟩

lemma pure_c:
  "pure φ ⟷ C φ = u"
  ⟨proof⟩

lemma smaller_pure:
  assumes "a << b"
      and "pure b"
    shows "pure a"
⟨proof⟩

lemma antisym:
  assumes "a << b"
      and "b << a"
    shows "a = b"
⟨proof⟩

lemma supported_inh:
  assumes "P φ"
      and "supported P"
    shows "{φ} >> Inh P"
⟨proof⟩

lemma supported_intui:
  assumes "supported P"
      and "intuitionistic P"
    shows "P φ ⟷ {φ} >> Inh P"
  ⟨proof⟩

lemma plus_and_bigger_set:
```

assumes *"A >> B"*
shows *"(A ⊕⊕ D) >> (B ⊕⊕ D)"*
⟨*proof*⟩

lemma *simple_set_add:*
assumes *"Some a = Some b ⊕ Some c"*
shows *"{b} ⊕⊕ {c} = {a}"* (**is** *"?s = ?a"*)
⟨*proof*⟩

lemma *set_add_elem:*
assumes *"x ∈ A ⊕⊕ B"*
shows *"∃ a b. a ∈ A ∧ b ∈ B ∧ Some x = Some a ⊕ Some b"*
⟨*proof*⟩

lemma *set_add_elem_reci:*
assumes *"a ∈ A ∧ b ∈ B ∧ Some x = Some a ⊕ Some b"*
shows *"x ∈ A ⊕⊕ B"*
⟨*proof*⟩

lemma *set_add_asso:*
"(A ⊕⊕ B) ⊕⊕ D = A ⊕⊕ (B ⊕⊕ D)" (**is** *"?g = ?d"*)
⟨*proof*⟩

lemma *bigger_set_union:*
assumes *"A' >> A"*
and *"B' >> B"*
shows *"A' ∪ B' >> A ∪ B"*
⟨*proof*⟩

lemma *smaller_core_comp:*
"b << a ⟷ |b| << |a| ∧ C b << C a" (**is** *"?a ⟷ ?b"*)
⟨*proof*⟩

lemma *c_trans_ineq:*
assumes *"b << a"*
and *"Some a = s_C x ⊕ Some aa"*
and *"Some b = s_C x ⊕ Some bb"*
shows *"bb << aa"*
⟨*proof*⟩

lemma *frame_trans:*
assumes *"b << a"*
and *"Some a = Some i ⊕ Some ra"*
and *"Some b = Some i ⊕ Some rb"*
and *"Some c = s_core i ⊕ Some ra"*
shows *"rb << c"*
⟨*proof*⟩

lemma *bigger_set_forall:*

```
  "A >> B ⟷ (∀ a∈A. {a} >> B)"
  ⟨proof⟩

lemma subset_smaller:
  assumes "A ⊆ B"
  shows "A >> B"
  ⟨proof⟩

lemma bigger_set_trans:
  assumes "A >> B"
      and "B >> D"
    shows "A >> D"
⟨proof⟩

lemma pure_set_add_smaller:
  assumes "A ⊕⊕ B = A"
  shows "A >> B"
  ⟨proof⟩

lemma decompose_set:
  assumes "x ∈ A ⊕⊕ B ⊕⊕ {c} ⊕⊕ {d}"
  shows "∃ a b. a ∈ A ∧ b ∈ B ∧ Some x = Some a ⊕ Some b ⊕ Some c ⊕
Some d"
  ⟨proof⟩

lemma add_pure_singleton:
  assumes "A >> B"
      and "A >> {p}"
      and "pure p"
    shows "A >> B ⊕⊕ {p}"
⟨proof⟩

lemma add_sets_neutral:
  "A ⊕⊕ {u} = A" (is "?a = ?b")
⟨proof⟩

end

end
```

A.3 Semantics

theory *Semantics*
  imports *SepAlgebra*
begin

A.3.1 Abstract language

```
datatype ('a, 'b, 'c) stmt =
  Inhale "'a assertion"
  | Assume "'a assertion"
  | Exhale "'a assertion"
  | Skip
  | Seq "('a, 'b, 'c) stmt" "('a, 'b, 'c) stmt" (infixl ";" 60)
  | If "('a, 'b, 'c) stmt" "('a, 'b, 'c) stmt"
  | Var "'b list"
  | Havoc "'b list"
  | MethodCall "'b list" "string" "'b list"
  | While "'a assertion" "'a assertion" "('a, 'b, 'c) stmt"
  | Other 'c

type_synonym ('a, 'b, 'c) method = "string × 'b list × 'b list × 'a
assertion × 'a assertion × ('a, 'b, 'c) stmt"

type_synonym ('a, 'b, 'c) program = "('a, 'b, 'c) method list"

fun get_method :: "('a, 'b, 'c) program ⇒ string ⇒ ('a, 'b, 'c) method
option" where
  "get_method (t # q) s = (if fst t = s then Some t else get_method q
s)"
| "get_method _ s = None"

fun get_args :: "('a, 'b, 'c) method ⇒ 'b list" where
  "get_args (_, args, _, _, _, _) = args"

fun get_ret :: "('a, 'b, 'c) method ⇒ 'b list" where
  "get_ret (_, _, ret, _, _, _) = ret"

fun get_args_ret :: "('a, 'b, 'c) method ⇒ 'b list" where
  "get_args_ret m = get_args m @ get_ret m"

fun get_pre :: "('a, 'b, 'c) method ⇒ 'a assertion" where
  "get_pre (_, _, _, P, _, _) = P"

fun get_post :: "('a, 'b, 'c) method ⇒ 'a assertion" where
  "get_post (_, _, _, _, Q, _) = Q"

fun get_body :: "('a, 'b, 'c) method ⇒ ('a, 'b, 'c) stmt" where
  "get_body (_, _, _, _, _, s) = s"
```

```
datatype 'a ss = S "'a set" | Error

type_synonym 'b rename_t = "'b list × 'b list × 'b list × 'b list"

fun get_S :: "'a ss ⇒ 'a set" where
  "get_S (S A) = A"
| "get_S _ = {}"

fun union_set_ss :: "'a ss set ⇒ 'a ss" where
  "union_set_ss A = (if Error ∈ A then Error
    else S (⋃a∈A. get_S a))"

fun len :: "('a, 'b, 'c) stmt ⇒ nat" where
  "len (MethodCall y m x) = 5"
| "len (While b I s) = 5 + len s"
| "len (If s1 s2) = len s1 + len s2"
| "len (s1 ; s2) = len s1 + len s2"
| "len _ = 1"

lemma len_at_least_one:
  "len s ≥ 1"
⟨proof⟩

definition lnot :: "'a assertion ⇒ 'a assertion" where
  "lnot P = (λs. ¬ P s)"

fun wf_renaming :: "'b rename_t ⇒ bool" where
  "wf_renaming (old_vars, new_vars, vars_to_avoid, domain) ⟷
      length old_vars = length new_vars ∧ distinct old_vars ∧ distinct
new_vars"

definition lfalse :: "'a assertion" where
  "lfalse = (λs. False)"

locale semantics_algebra = sep_algebra +

  fixes σ :: "'a ⇒ 'b set"

  fixes semantics_other :: "('a, 'b, 'c) program ⇒ 'a ⇒ 'c ⇒ 'a ss"

  fixes read_pred :: "'a assertion ⇒ 'b list"
  fixes read_other :: "'c ⇒ 'b list"

  fixes well_defined_assert :: "'a assertion ⇒ 'a ⇒ bool"

  fixes modif_other :: "'c ⇒ 'b list"

  fixes rename_elem :: "'b ⇒ 'b rename_t ⇒ 'b"
  fixes rename_other :: "'c ⇒ 'b rename_t ⇒ 'c"
```

```
  fixes rename_state :: "'a ⇒ 'b rename_t ⇒ 'a"
  fixes rename_pred :: "'a assertion ⇒ 'b rename_t ⇒ 'a assertion"
  fixes rename_inv :: "'b rename_t ⇒ 'b rename_t"

  fixes wf_other :: "('a, 'b, 'c) program ⇒ 'c ⇒ bool"

assumes

      store_pure: "σ φ = σ |φ|"
  and store_add: "Some a = Some c ⊕ Some d ⟹ σ a = σ c ∪ σ d"
  and store_empty: "σ u = {}"
  and compatible_stores: "a ## b ⟹ pure a ⟹ σ a ⊆ σ b ⟹ a << b"
  and unique_store_exists: "var ∈ σ a ⟹ (∃c. σ c = {var} ∧ c << a)"
  and defined_disjoint_store: "σ a ∩ σ b = {} ⟹ pure a ⟹ a ## b"

  and read_pred_def: "set (read_pred P) = (⋃i∈Inh P. σ i)"
  and well_defined_assert_supported: "supported P ⟹ (well_defined_assert
P φ ⟷ set (read_pred P) ⊆ σ φ)"
  and can_read_not: "well_defined_assert P a ⟷ well_defined_assert
(lnot P) a"
  and p_implies_well_def: "P φ ⟹ well_defined_assert P φ"

  and modif_other_sem: "wf_other Pr other ∧ c ∈ get_S (semantics_other
Pr a other) ⟹ σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif_other other)"
  and modif_other_read_other: "set (modif_other other) ⊆ set (read_other
other)"

  and rename_inv_def_elem: "wf_renaming t ⟹ rename_elem (rename_elem
elem t) (rename_inv t) = elem"
  and rename_inv_def_other: "wf_renaming t ⟹ rename_other (rename_other
other t) (rename_inv t) = other"
  and renaming_invert_wf: "wf_renaming t ⟹ wf_renaming (rename_inv
t)"

  and rename_removes_vars_elem: "wf_renaming (l1, l2, l3, do) ⟹ var
∈ set do ⟹
           rename_elem var (l1, l2, l3, do) ∈ set l1 ∪ set l3 ⟹ rename_elem
var (l1, l2, l3, do) ∈ set l2"
  and rename_elem_in_l1_l2: "wf_renaming (l1, l2, l3, do) ⟹ i < length
l1 ⟹ rename_elem (l1 ! i) (l1, l2, l3, do) = l2 ! i"

  and rename_store: "wf_renaming t ⟹ σ (rename_state a t) = (λelem.
rename_elem elem t) ‘ (σ a)"
  and rename_state_add: "wf_renaming t ⟹ Some a = Some d ⊕ Some c ⟹
Some (rename_state a t) = Some (rename_state d t) ⊕ Some (rename_state
c t)"
  and rename_state_identity: "wf_renaming t ⟹ (∀x∈σ a. rename_elem
x t = x) ⟹ rename_state a t = a"
  and rename_state_double: "wf_renaming t ⟹ wf_renaming t1 ⟹ wf_renaming
```

```
t2 ⟹
            (∀x∈σ a. rename_elem x t = rename_elem (rename_elem x t1)
t2) ⟹ rename_state a t = rename_state (rename_state a t1) t2"

  and rename_pred_def: "rename_pred P t = (λφ. P (rename_state φ (rename_inv
t)))"
  and well_defined_assert_rename: "wf_renaming t ⟹ well_defined_assert
P φ ⟹ well_defined_assert (rename_pred P t) (rename_state φ t)"

  and read_other_rename_other: "read_other (rename_other other t) = map
(λelem. rename elem t) (read_other other)"
  and modif_rename_other: "modif_other (rename_other other t) = map (λelem.
rename_elem elem t) (modif_other other)"
  and wf_other_renaming: "wf_renaming t ⟹ wf_other Pr other ⟹ wf_other
Pr (rename_other other t)"
  and ver_rename_other: "wf_renaming t ⟹ semantics_other Pr φ other
≠ Error  ⟹ semantics_other Pr (rename_state φ t) (rename_other other
t) ≠ Error"
  and semantics_rename_other: "wf_renaming t ⟹ a ∈ get_S (semantics_other
Pr (rename_state φ t) (rename_other other t))
                                ⟹ rename_state a (rename_inv t) ∈ get_S
(semantics_other Pr φ other)"

begin

lemma pure_is_exactly_store: "a << c ⟹ b << c ⟹ pure a ⟹ σ a ⊆
σ b ⟹ a << b"
  ⟨proof⟩

lemma rename_state_c_same:
  assumes "wf_renaming t"
  shows "C (rename_state a t) = C a"
⟨proof⟩

lemma finite_store:
  "finite (σ x)"
⟨proof⟩

lemma rename_state_same:
  assumes "wf_renaming t1"
      and "wf_renaming t2"
      and "⋀x. x ∈ σ a ⟹ rename_elem x t1 = rename_elem x t2"
    shows "rename_state a t1 = rename_state a t2"
⟨proof⟩

lemma rename_inv_def_state:
  assumes "wf_renaming t"
    shows "rename_state (rename_state a t) (rename_inv t) = a" (is "?a
= a")
```

⟨*proof*⟩

lemma *rename_store_same_if_l1:*
  assumes *"wf_renaming (l1, l2, l3, do)"*
      and *"σ a ⊆ set l1"*
    shows *"rename_state a (l1, l2, l3, do) = rename_state a (l1, l2, [], d)"*
⟨*proof*⟩

definition *read_mono :: "'a assertion ⇒ bool"* **where**
  "read_mono P ⟷ (∀ φ φ'. φ << φ' ⟶ well_defined_assert P φ ⟶ well_defined_assert P φ')"

definition *wf_assert :: "'a assertion ⇒ bool"* **where**
  "wf_assert P ⟷ supported P ∧ intuitionistic P ∧ read_mono P"

lemma *can_read_false:*
  "well_defined_assert lfalse a"
⟨*proof*⟩

fun *rename_set :: "'a set ⇒ 'b rename_t ⇒ 'a set"* **where**
  "rename_set A t = (λφ. rename_state φ t) ‘ A"

lemma *rename_pred_same:*
  assumes *"wf_renaming t"*
  shows *"P φ ⟷ (rename_pred P t) (rename_state φ t)"*
  ⟨*proof*⟩

lemma *wf_renaming_order:*
  assumes *"wf_renaming t"*
    and *"a << b"*
  shows *"rename_state a t << rename_state b t"*
  ⟨*proof*⟩

lemma *rename_doesnt_change_if_not_affected:*
  assumes *"wf_renaming (l1, l2, [], do)"*
      and *"set (read_pred P) ⊆ set l1"*
      and *"wf_assert P"*
    shows *"rename_pred P (l1, l2, [], do) = rename_pred P (l1, l2, l, d)"*
⟨*proof*⟩

lemma *wf_renaming_diff:*
  assumes *"wf_renaming t"*
    and *"a ≠ b"*
  shows *"rename_state a t ≠ rename_state b t"*
  ⟨*proof*⟩

lemma *inh_renamed:*

```
  assumes "wf_renaming t"
  shows "Inh (rename_pred P t) = rename_set (Inh P) t" (is "?a = ?b")
⟨proof⟩

lemma read_pred_rename_pred:
  assumes "wf_renaming t"
  shows "set (read_pred (rename_pred P t)) = (λelem. rename_elem elem
t) ‘ (set (read_pred P))" (is "?a = ?b")
⟨proof⟩

lemma wf_rename_inv_other:
  assumes "wf_renaming t"
  shows "rename_state (rename_state x (rename_inv t)) t = x"
  ⟨proof⟩

definition rename_list :: "'b list ⇒ 'b rename_t ⇒ 'b list" where
  "rename_list l t = map (λelem. rename_elem elem t) l"

lemma rename_pred_comp_simple:
  assumes "wf_renaming t"
      and "wf_renaming (l1, l2, [], do)"
      and "set (read_pred P) ⊆ set l1"
      and "wf_assert P"
    shows "rename_pred P (l1, rename_list l2 t, [], do) = rename_pred
(rename_pred P (l1, l2, [], do)) t" (is "?ra = ?rb")
⟨proof⟩

lemma rename_state_neutral:
  assumes "wf_renaming t"
  shows "rename_state u t = u" (is "?r = u")
⟨proof⟩

lemma rename_removes_vars_other:
  assumes "wf_renaming (l1, l2, l3, do)"
      and "set (read_other other) ⊆ set do"
    shows "set (read_other (rename_other other (l1, l2, l3, do))) ∩ (set
l1 ∪ set l3) ⊆ set l2" (is "?a ⊆ ?b")
⟨proof⟩

lemma pure_is_exactly_store_variant:
  assumes "a << b"
      and "pure b"
      and "σ a = σ b"
    shows "a = b"
⟨proof⟩

definition h :: "'b list ⇒ 'a set" where
  "h V = {φ. pure φ ∧ σ φ = set V}"
```

lemma *h_store:*
  assumes *"φ ∈ h l"*
  shows *"σ φ = set l"*
  ⟨*proof*⟩

lemma *h_pure:*
  assumes *"hx ∈ h x"*
  shows *"pure hx"*
  ⟨*proof*⟩

lemma *all_stores_exist:*
  "s ⊆ σ a ⟹ ∃c. c << a ∧ σ c = s"
⟨*proof*⟩

lemma *h_bigger:*
  assumes *"set x ⊆ σ φ"*
  shows *"{φ} >> h x"*
⟨*proof*⟩

lemma *h_lin_simpler:*
  assumes *"set v = set v1 ∪ set v2"*
      and *"set v1 ∩ set v2 = {}"*
  shows *"h v = h v1 ⊕⊕ h v2"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *h_v_add:*
  "h v = h v ⊕⊕ h v" (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *exists_list_inter:*
  "⋀b. ∃c. set c = set a ∩ set b"
⟨*proof*⟩

lemma *exists_list_minus:*
  "⋀b. ∃c. set c = set a - set b"
⟨*proof*⟩

lemma *h_lin:*
  assumes *"set v = set v1 ∪ set v2"*
  shows *"h v = h v1 ⊕⊕ h v2"* (**is** *"?a = ?b"*)
⟨*proof*⟩

fun *h_comp_some :: "'a ⇒ 'b list ⇒ 'a"* **where**
  "h_comp_some a l = (THE b. b << a ∧ σ b = σ a - set l ∧ pure b)"

lemma *h_comp_some_exists:*
  assumes *"b = h_comp_some a l"*
  shows *"b << a ∧ σ b = σ a - set l ∧ pure b"*
⟨*proof*⟩

definition *h_comp :: "'a ⇒ 'b list ⇒ 'a"* **where**
"h_comp φ V = (the (s_C φ ⊕ Some (h_comp_some φ V)))"

lemma *h_comp_some_sum:*
"Some (h_comp φ V) = s_C φ ⊕ Some (h_comp_some φ V)"
⟨*proof*⟩

lemma *h_comp_store:*
"σ (h_comp φ l) = σ φ - set l"
⟨*proof*⟩

lemma *h_comp_grows:*
assumes *"c << a"*
shows *"h_comp c V << h_comp a V"*
⟨*proof*⟩

lemma *h_comp_lin:*
assumes *"Some a = Some a1 ⊕ Some a2"*
shows *"Some (h_comp a V) = Some (h_comp a1 V) ⊕ Some (h_comp a2 V)"*
⟨*proof*⟩

lemma *h_comp_not_here:*
assumes *"set V ∩ σ a = {}"*
shows *"h_comp a V = a"*
⟨*proof*⟩

lemma *h_comp_sum:*
"Some φ = Some (h_comp φ x) ⊕ s_core φ"
⟨*proof*⟩

lemma *h_comp_h:*
assumes *"φ ∈ h V"*
shows *"h_comp φ V = u"*
⟨*proof*⟩

lemma *rename_removes_vars_list:*
assumes *"wf_renaming (l1, l2, l3, do)"*
and *"set l ⊆ set do"*
shows *"set (rename_list l (l1, l2, l3, do)) ∩ (set l1 ∪ set l3) ⊆ set l2"* (**is** *"?a ⊆ ?b"*)
⟨*proof*⟩

lemma *list_inclusion:*
assumes *"wf_renaming t"*
and *"set a ⊆ set b"*
shows *"set (rename_list a t) ⊆ set (rename_list b t)"* (**is** *"?a ⊆ ?b"*)
⟨*proof*⟩

lemma *rename_list_same:*
assumes *"wf_renaming (l1, l2, l3, do)"*
shows *"rename_list l1 (l1, l2, l3, do) = l2"*
⟨*proof*⟩

lemma *rename_modif_for_q:*
assumes *"wf_renaming (l1, l2, l3, do)"*
and *"set (read_pred Q) ⊆ set l1"*
shows *"set (read_pred (rename_pred Q (l1, l2, l3, do))) ⊆ set l2"*
(is *"?a ⊆ ?b")*
⟨*proof*⟩

lemma *rename_removes_vars_pred:*
assumes *"wf_renaming (l1, l2, l3, do)"*
and *"set (read_pred P) ⊆ set do"*
shows *"set (read_pred (rename_pred P (l1, l2, l3, do))) ∩ (set l1 ∪ set l3) ⊆ set l2"*
⟨*proof*⟩

lemma *rename_inv_def_pred:*
assumes *"wf_renaming t"*
shows *"rename_pred (rename_pred P t) (rename_inv t) = P"*
⟨*proof*⟩

lemma *rename_pred_lnot:*
"rename_pred (lnot b) t = lnot (rename_pred b t)"
⟨*proof*⟩

lemma *rename_list_same_length: "wf_renaming t ⟹ length (rename_list l t) = length l"*
⟨*proof*⟩

lemma *rename_list_concat: "wf_renaming t ⟹ rename_list (l1 @ l2) t = rename_list l1 t @ rename_list l2 t"*
⟨*proof*⟩

lemma *rename_elem_list:*
assumes *"i < length l"*
and *"wf_renaming t"*
shows *"rename_list l t ! i = rename_elem (l ! i) t"*
⟨*proof*⟩

lemma *rename_list_distinct:*
assumes *"wf_renaming t"*
and *"distinct l"*
shows *"distinct (rename_list l t)"*
⟨*proof*⟩

fun *modif :: "('a, 'b, 'c) stmt ⇒ 'b list"* **where**

```
  "modif (s1 ; s2) = modif s1 @ modif s2"
| "modif (If s1 s2) = modif s1 @ modif s2"
| "modif (MethodCall y m x) = y"
| "modif (While b I s) = modif s"
| "modif (Assume b) = []"
| "modif (Inhale P) = []"
| "modif (Exhale P) = []"
| "modif (Var l) = l"
| "modif (Havoc l) = l"
| "modif Skip = []"
| "modif (Other s) = modif_other s"

fun sigma_list :: "'a ⇒ 'b list" where
  "sigma_list φ = (SOME l. set l = σ φ)"

lemma sigma_list_def:
  "set (sigma_list φ) = σ φ"
  ⟨proof⟩
```

A.3.2 Semantics

```
function semantics :: "('a, 'b, 'c) program ⇒ 'a ⇒ ('a, 'b, 'c) stmt
⇒ 'a ss" where
  "semantics Pr φ Skip = S {φ}"
| "semantics Pr φ (Assume b) = (if well_defined_assert b φ then if b
φ then S {φ} else S {}
  else Error)"

| "semantics Pr φ (s1 ; s2) = (let r = semantics Pr φ s1 in
  if r = Error then Error
  else let A = get_S r in
    union_set_ss ((λφ'. semantics Pr φ' s2) ‘ A))"

| "semantics Pr φ (If s1 s2) = (let r1 = semantics Pr φ s1 in
    let r2 = semantics Pr φ s2 in
    if r1 = Error ∨ r2 = Error then Error
    else S (get_S r1 ∪ get_S r2))"

| "semantics Pr φ (Var x) = (if set x ∩ σ φ = {} then S ({φ} ⊕⊕ h x)
else Error)"
| "semantics Pr φ (Other s) = semantics_other Pr φ s"
| "semantics Pr φ (Havoc x) = (if set x ⊆ σ φ then S ({h_comp φ x} ⊕⊕
h x) else Error)"

| "semantics Pr φ (Inhale P) = (if well_defined_assert P φ then S ({φ}
⊕⊕ Inh P) else Error)"
| "semantics Pr φ (Exhale P) =
  (if P φ ∧ well_defined_assert P φ then S {φ' | φ' i r. Some φ = Some
i ⊕ Some r ∧ i ∈ Inh P ∧ Some φ' = s_core i ⊕ Some r}
```

```
  else Error)"

| "semantics Pr φ (MethodCall y m x) = (
  if set x ∪ set y ⊆ σ φ then
    let (_, args, ret, P, Q, _) = the (get_method Pr m) in semantics Pr
φ
    (Exhale (rename_pred P (args @ ret, x @ y, [], [])); Havoc y ; Inhale
(rename_pred Q (args @ ret, x @ y, [], [])))
  else Error)"
| "semantics Pr φ (While b I s) =
   (let V = modif s in
   if semantics Pr ( |φ| ) (Havoc V; Inhale I; Assume b; s ; Exhale I)
= Error then
     Error
   else
     semantics Pr φ (Exhale I; Havoc V; Inhale I ; Assume (lnot b)))"
```
⟨*proof*⟩
termination
⟨*proof*⟩

```
fun read :: "('a, 'b, 'c) stmt ⇒ 'b list" where
  "read (s1 ; s2) = read s1 @ read s2"
| "read (If s1 s2) = read s1 @ read s2"
| "read (MethodCall y m x) = x @ y"
| "read (While b I s) = read s @ read_pred I @ read_pred b"
| "read (Assume b) = read_pred b"
| "read (Inhale P) = read_pred P"
| "read (Exhale P) = read_pred P"
| "read (Var l) = l"
| "read (Havoc l) = l"
| "read Skip = []"
| "read (Other s) = read_other s"

fun rename :: "('a, 'b, 'c) stmt ⇒ 'b rename_t ⇒ ('a, 'b, 'c) stmt"
where
  "rename (Inhale P) t = Inhale (rename_pred P t)"
| "rename (Exhale P) t = Exhale (rename_pred P t)"
| "rename (s1 ; s2) t = (rename s1 t ; rename s2 t)"
| "rename (Var x) t = Var (rename_list x t)"
| "rename (Havoc x) t = Havoc (rename_list x t)"
| "rename (If s1 s2) t = If (rename s1 t) (rename s2 t)"
| "rename (While b I s) t = While (rename_pred b t) (rename_pred I t)
(rename s t)"
| "rename (MethodCall y m x) t = MethodCall (rename_list y t) m (rename_list
x t)"
| "rename (Assume b) t = Assume (rename_pred b t)"
| "rename (Other other) t = Other (rename_other other t)"
| "rename Skip t = Skip"
```

```
fun bigger_ss :: "'a ss ⇒ 'a ss ⇒ bool" (infixl ">>>" 60) where
  "_ >>> Error ⟷ True"
| "Error >>> _ ⟷ False"
| "S A >>> S B ⟷ A >> B"

fun sem :: "('a, 'b, 'c) program ⇒ 'a set ⇒ ('a, 'b, 'c) stmt ⇒ 'a
set" where
  "sem Pr A s = (⋃a∈A. get_S (semantics Pr a s))"

definition ver :: "('a, 'b, 'c) program ⇒ 'a set ⇒ ('a, 'b, 'c) stmt
⇒ bool" where
  "ver Pr A st ⟷ (∀a∈A. semantics Pr a st ≠ Error)"

definition mono :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt ⇒ bool"
where
  "mono Pr s ⟷ (∀a b. b << a ⟶ semantics Pr a s >>> semantics Pr
b s)"

definition smonoIn :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt  ⇒ bool"
where
  "smonoIn Pr st ⟷ (∀A B. A >> B ⟶ (ver Pr B st ⟶ ver Pr A st))"

definition smonoOut :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt ⇒ bool"
where
  "smonoOut Pr st ⟷ (∀A B. ver Pr B st ∧ A >> B ⟶ (sem Pr A st >>
sem Pr B st))"

definition smono :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt ⇒ bool"
where
  "smono Pr st ⟷ smonoIn Pr st ∧ smonoOut Pr st"

lemma v_singleton: "ver Pr A s ⟷ (∀a∈A. ver Pr {a} s)"
  ⟨proof⟩

lemma s_singleton: "sem Pr A s = (⋃a∈A. sem Pr {a} s)"
  ⟨proof⟩

lemma elem_sem:
  "x ∈ sem Pr A s ⟷ (∃a∈A. x ∈ sem Pr {a} s)"
  ⟨proof⟩

lemma smaller_error:
  assumes "Error >>> x"
  shows "x = Error"
  ⟨proof⟩

lemma bigger_not_error:
  assumes "x >>> y"
      and "y ≠ Error"
```

```
    shows "x ≠ Error"
  ⟨proof⟩

lemma smonoIn_singleton:
  "smonoIn Pr s ⟷ (∀ a b. b << a ⟶ (ver Pr {b} s ⟶ ver Pr {a} s))"
(is "?a ⟷ ?b")
⟨proof⟩

lemma smonoOut_singleton:
  "smonoOut Pr s ⟷ (∀ a b. ver Pr {b} s ∧ b << a ⟶ (sem Pr {a} s
>> sem Pr {b} s))" (is "?a ⟷ ?b")
⟨proof⟩

lemma mono_smono:
  "mono Pr s ⟷ smono Pr s" (is "?a ⟷ ?b")
⟨proof⟩

fun wf_stmt :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt ⇒ bool" where
  "wf_stmt Pr (s1; s2) ⟷ wf_stmt Pr s1 ∧ wf_stmt Pr s2"
| "wf_stmt Pr (If s1 s2) ⟷ wf_stmt Pr s1 ∧ wf_stmt Pr s2"
| "wf_stmt Pr (While b I s) ⟷ wf_assert I ∧ wf_stmt Pr s"
| "wf_stmt Pr (MethodCall y m x) ⟷ (let r = get_method Pr m in
  r ≠ None ∧ (let (_, args, ret, _, _, _) = the r in length x = length
args
  ∧ length y = length ret ∧ distinct (x @ y)))"
| "wf_stmt Pr (Inhale P) ⟷ supported P"
| "wf_stmt Pr (Exhale P) ⟷ supported P"
| "wf_stmt Pr (Other other) ⟷ wf_other Pr other"
| "wf_stmt Pr _ ⟷ True"

fun wf_method :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) method ⇒ bool"
where
  "wf_method Pr (m, args, ret, P, Q, s) ⟷
    wf_assert P ∧ wf_assert Q ∧ distinct (args @ ret) ∧ set args ∩ set
(modif s) = {} ∧
    set (read_pred P) ⊆ set args ∧ wf_stmt Pr s ∧ set (read_pred Q) ⊆
set args ∪ set ret"

fun wf_program_aux :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) program ⇒
bool" where
  "wf_program_aux Pr [] ⟷ True"
| "wf_program_aux Pr (t # q) ⟷ wf_method Pr t ∧ wf_program_aux Pr q"

fun wf_program :: "('a, 'b, 'c) program ⇒ bool" where
  "wf_program Pr ⟷ wf_program_aux Pr Pr"

lemma get_method_same_name:
  "get_method Pr m ≠ None ⟹ fst (the (get_method Pr m)) = m"
  ⟨proof⟩
```

lemma *simple_method_exists:*
assumes *"wf_stmt Pr (MethodCall y m x)"*
shows *"∃ args ret P Q s. get_method Pr m = Some (m, args, ret, P, Q, s)"*
⟨*proof*⟩

lemma *sem_loop:*
assumes *"ver Pr {φ} (While b I s)"*
shows *"sem Pr {φ} (While b I s) = sem Pr {φ} (Exhale I; Havoc (modif s); Inhale I; Assume (lnot b))"*
⟨*proof*⟩

lemma *ver_method_real:*
assumes *"get_method Pr m = Some (m, args, ret, P, Q, s)"*
shows *"ver Pr {φ} (MethodCall y m x) ⟷*
ver Pr {φ} (Exhale (rename_pred P (args @ ret, x @ y, [], [])) ; Havoc y ; Inhale (rename_pred Q (args @ ret, x @ y, [], []))) ∧
(set x ∪ set y) ⊆ σ φ"
⟨*proof*⟩

lemma *ver_loop: "ver Pr {φ} (While b I s) ⟷*
ver Pr {φ} (Exhale I; Havoc (modif s); Inhale I ; Assume (lnot b)) ∧
ver Pr { |φ| } (Havoc (modif s); Inhale I; Assume b; s ; Exhale I)"
⟨*proof*⟩

lemma *sem_method_real:*
assumes *"get_method Pr m = Some (m, args, ret, P, Q, s)"*
and *"ver Pr {φ} (MethodCall y m x)"*
shows *"sem Pr {φ} (MethodCall y m x) =*
sem Pr {φ} (Exhale (rename_pred P (args @ ret, x @ y, [], [])); Havoc y ;
Inhale (rename_pred Q (args @ ret, x @ y, [], [])))"
⟨*proof*⟩

definition *well_defined_assert_set :: "'a assertion ⇒ 'a set ⇒ bool"* **where**
"well_defined_assert_set P A ⟷ (∀ a∈A. well_defined_assert P a)"

lemma *ver_inhale_single:*
"well_defined_assert P φ ⟷ ver Pr {φ} (Inhale P)"
⟨*proof*⟩

lemma *ver_inhale:*
"well_defined_assert_set P A ⟷ ver Pr A (Inhale P)"
⟨*proof*⟩

lemma *union_sum:*
"(⋃ a∈A. {a} ⊕⊕ B) = A ⊕⊕ B" (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *sem_inhale:*
assumes *"well_defined_assert_set P A"*
shows *"sem Pr A (Inhale P) = A ⊕⊕ Inh P"*
⟨*proof*⟩

lemma *ver_exhale:*
"ver Pr {φ} (Exhale P) = P φ"
⟨*proof*⟩

lemma *singleton_sem:*
assumes *"ver Pr {φ} s"*
shows *"sem Pr {φ} s = get_S (semantics Pr φ s)"*
⟨*proof*⟩

lemma *sem_exhale:*
assumes *"P φ"*
shows *"sem Pr {φ} (Exhale P) = {φ' | φ' i r. Some φ = Some i ⊕ Some r ∧ i ∈ Inh P ∧ Some φ' = s_core i ⊕ Some r}"*
⟨*proof*⟩

lemma *sem_skip:*
"(ver Pr A Skip) ∧ (sem Pr A Skip = A)"
⟨*proof*⟩

lemma *ver_if:*
"ver Pr A (If s1 s2) = (ver Pr A s1 ∧ ver Pr A s2)"
⟨*proof*⟩

lemma *sem_if_single:*
assumes *"ver Pr {φ} (If s1 s2)"*
shows *"sem Pr {φ} (If s1 s2) = sem Pr {φ} s1 ∪ sem Pr {φ} s2"*
⟨*proof*⟩

lemma *sem_if:*
assumes *"ver Pr A (If s1 s2)"*
shows *"sem Pr A (If s1 s2) = sem Pr A s1 ∪ sem Pr A s2"*
⟨*proof*⟩

lemma *sem_assume_true:*
assumes *"b φ"*
and *"well_defined_assert b φ"*
shows *"sem Pr {φ} (Assume b) = {φ}"*
⟨*proof*⟩

lemma *sem_assume_false:*
assumes *"¬ b φ"*
shows *"sem Pr {φ} (Assume b) = {}"*
⟨*proof*⟩

lemma *ver_havoc:*
"ver Pr {φ} (Havoc l) ⟷ set l ⊆ σ φ"
⟨*proof*⟩

lemma *sem_havoc:*
assumes *"ver Pr {φ} (Havoc l)"*
shows *"sem Pr {φ} (Havoc l) = {h_comp φ l} ⊕⊕ h l"*
⟨*proof*⟩

lemma *ver_var:*
"ver Pr {φ} (Var x) ⟷ set x ∩ σ φ = {}"
⟨*proof*⟩

lemma *sem_var:*
assumes *"ver Pr A (Var x)"*
shows *"sem Pr A (Var x) = A ⊕⊕ h x"*
⟨*proof*⟩

lemma *ver_seq_single:*
"ver Pr {φ} (Seq s1 s2) ⟷ ver Pr {φ} s1 ∧ ver Pr (sem Pr {φ} s1) s2" (**is** *"?a ⟷ ?b"*)
⟨*proof*⟩

lemma *ver_seq:*
"ver Pr A (Seq s1 s2) ⟷ ver Pr A s1 ∧ ver Pr (sem Pr A s1) s2"
⟨*proof*⟩

lemma *union_ss:*
assumes *"union_set_ss ((λφ'. semantics Pr φ' s) ‘ A) = S r"*
and *"ver Pr A s"*
shows *"r = sem Pr A s"*
⟨*proof*⟩

lemma *sem_seq_single:*
assumes *"ver Pr {φ} (s1 ; s2)"*
shows *"sem Pr {φ} (Seq s1 s2) = sem Pr (sem Pr {φ} s1) s2"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *sem_seq:*
assumes *"ver Pr A (s1 ; s2)"*
shows *"sem Pr A (Seq s1 s2) = sem Pr (sem Pr A s1) s2"*
⟨*proof*⟩

lemma *rename_set_empty:*
assumes *"wf_renaming t"*
shows *"rename_set {u} t = {u}"* (**is** *"?a = ?b"*)
⟨*proof*⟩

```
lemma store_same_pred_supp_inhale:
  assumes "supported P"
      and "ver Pr {a} (Inhale P)"
      and "b ∈ sem Pr {a} (Inhale P)"
    shows "σ b = σ a"
⟨proof⟩

lemma exhale_verif:
  assumes "ver Pr {φ} (Exhale P)"
    shows "P φ"
  ⟨proof⟩

lemma inh_and_supported:
  assumes "supported P"
      and "a ∈ Inh P"
      and "b ∈ Inh P"
      and "a << φ"
      and "b << φ"
      and "P φ"
    shows "a = b"
  ⟨proof⟩

lemma exhale_sem_inh:
  assumes "supported P"
      and "i ∈ Inh P"
      and "Some φ = Some i ⊕ Some r"
      and "ver Pr {φ} (Exhale P)"
    shows "sem Pr {φ} (Exhale P) = {the (s_core i ⊕ Some r)}"
⟨proof⟩

lemma store_same_pred_supp_exhale:
  assumes "supported P"
      and "ver Pr {a} (Exhale P)"
      and "b ∈ sem Pr {a} (Exhale P)"
    shows "σ b = σ a"
⟨proof⟩

lemma modif_in_read: "set (modif s) ⊆ set (read s)"
⟨proof⟩

lemma rename_modif_list:
  assumes "wf_renaming t"
  shows "modif (rename s t) = rename_list (modif s) t"
⟨proof⟩

lemma rename_modif_no_inter_elem: "wf_renaming t ∧ set l ∩ set (modif
s) = {}
          ⟹ set (rename_list l t) ∩ set (modif (rename s t)) = {}"
```

(**is** *"?a ⟹ ?b"*)
⟨*proof*⟩

lemma *rename_store_inv: "wf_renaming t ∧ set (modif (rename s t)) ∩ σ (rename_state r t) = {} ⟹ set (modif s) ∩ σ r = {}"* (**is** *"?a ⟹ ?b"*)
⟨*proof*⟩

lemma *rename_modif_no_inter:*
assumes *"wf_renaming (args @ ret, x @ y, l, do)"*
and *"length args = length x"*
and *"set args ∩ set (modif s) = {}"*
and *"set (modif s) ⊆ set do"*
shows *"set x ∩ set (modif (rename s (args @ ret, x @ y, l, do))) = {}"*
⟨*proof*⟩

lemma *rename_store_lemma: "wf_renaming t ⟹ set (modif (rename s t)) ∩ σ r = {} ⟹ set (modif s) ∩ σ (rename_state r (rename_inv t)) = {}"*
⟨*proof*⟩

lemma *rename_removes_vars:*
"wf_renaming (l1, l2, l3, do) ∧ set (read s) ⊆ set do ⟶ set (read (rename s (l1, l2, l3, do))) ∩ (set l1 ∪ set l3) ⊆ set l2"
⟨*proof*⟩

lemma *rename_list_inv:*
assumes *"wf_renaming t"*
shows *"rename_list (rename_list l t) (rename_inv t) = l"*
⟨*proof*⟩

lemma *renaming_invert:*
"wf_renaming t ⟶ rename (rename s t) (rename_inv t) = s"
⟨*proof*⟩

lemma *member_rename_set_inv:*
assumes *"wf_renaming t"*
shows *"rename_state x (rename_inv t) ∈ A ⟷ x ∈ rename_set A t"*
⟨*proof*⟩

lemma *h_rename_set:*
assumes *"wf_renaming t"*
shows *"h (rename_list V t) = rename_set (h V) t"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *rename_set_add:*
assumes *"wf_renaming t"*
shows *"rename_set (A ⊕⊕ B) t = rename_set A t ⊕⊕ rename_set B t"*
(**is** *"?A = ?B"*)
⟨*proof*⟩

lemma *h_comp_c_same:*
"C (h_comp x l) = C x"
⟨*proof*⟩

lemma *wf_renaming_rename_list_set:*
assumes *"wf_renaming t"*
and *"set A = set B - set D"*
shows *"set (rename_list A t) = set (rename_list B t) - set (rename_list D t)"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *h_comp_rename:*
assumes *"wf_renaming t"*
shows *"h_comp (rename_state φ t) (rename_list V t) = rename_state (h_comp φ V) t"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *wf_core_rename_same:*
assumes *"wf_renaming t"*
shows *"rename_state (|x|) t = | rename_state x t |"*
⟨*proof*⟩

definition *rename_sem_property :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt ⇒ 'b rename_t ⇒ bool"* **where**
"rename_sem_property Pr s t ⟷ (∀ φ. wf_renaming t ∧ wf_stmt Pr s ∧ wf_program Pr ∧ ver Pr {φ} s ⟶
ver Pr {rename_state φ t} (rename s t) ∧ sem Pr {rename_state φ t} (rename s t) = rename_set (sem Pr {φ} s) t)"

lemma *rename_sem_inhale:*
assumes *"wf_renaming t"*
and *"ver Pr {φ} (Inhale P)"*
shows *"ver Pr {rename_state φ t} (rename (Inhale P) t)"*
and *"sem Pr {rename_state φ t} (rename (Inhale P) t) = rename_set (sem Pr {φ} (Inhale P)) t"*
⟨*proof*⟩

lemma *rename_sem_assume:*
assumes *"wf_renaming t"*
and *"ver Pr {φ} (Assume b)"*
shows *"ver Pr {rename_state φ t} (rename (Assume b) t)"*
and *"sem Pr {rename_state φ t} (rename (Assume b) t) = rename_set (sem Pr {φ} (Assume b)) t"*
⟨*proof*⟩

lemma *rename_sem_exhale:*
assumes *"wf_renaming t"*
and *"ver Pr {φ} (Exhale P)"*

```
    shows "ver Pr {rename_state φ t} (rename (Exhale P) t)"
      and "sem Pr {rename_state φ t} (rename (Exhale P) t) = rename_set
(sem Pr {φ} (Exhale P)) t"
⟨proof⟩

lemma rename_sem_havoc:
  assumes "wf_renaming t"
      and "ver Pr {φ} (Havoc V)"
    shows "ver Pr {rename_state φ t} (rename (Havoc V) t)"
      and "sem Pr {rename_state φ t} (rename (Havoc V) t) = rename_set
(sem Pr {φ} (Havoc V)) t"
⟨proof⟩

lemma rename_sem_var:
  assumes "wf_renaming t"
      and "ver Pr {φ} (Var V)"
    shows "ver Pr {rename_state φ t} (rename (Var V) t)"
      and "sem Pr {rename_state φ t} (rename (Var V) t) = rename_set
(sem Pr {φ} (Var V)) t"
⟨proof⟩

lemma rename_sem_seq:
  assumes "wf_renaming t"
      and "ver Pr {φ} (s1 ; s2)"
      and "wf_stmt Pr (s1 ; s2)"
      and "wf_program Pr"
      and "⋀φ. wf_stmt Pr s1 ⟹ wf_program Pr ⟹ ver Pr {φ} s1 ⟹
ver Pr {rename_state φ t} (rename s1 t) ∧ sem Pr {rename_state φ t} (rename
s1 t) = rename_set (sem Pr {φ} s1) t"
      and "⋀φ. wf_stmt Pr s2 ⟹ wf_program Pr ⟹ ver Pr {φ} s2 ⟹
ver Pr {rename_state φ t} (rename s2 t) ∧ sem Pr {rename_state φ t} (rename
s2 t) = rename_set (sem Pr {φ} s2) t"
    shows "ver Pr {rename_state φ t} (rename (s1 ; s2) t)"
      and "sem Pr {rename_state φ t} (rename (s1 ; s2) t) = rename_set
(sem Pr {φ} (s1 ; s2)) t"
⟨proof⟩

lemma seq_rename_sem_property:
  assumes "rename_sem_property Pr s1 t"
      and "rename_sem_property Pr s2 t"
      and "wf_renaming t"
    shows "rename_sem_property Pr (s1 ; s2) t"
⟨proof⟩

lemma wf_program_method_aux:
  "wf_program_aux Pr P ∧ get_method P name = Some m ⟹ wf_method Pr
m"
⟨proof⟩
```

lemma *wf_method_exists_equiv:*
assumes *"get_method Pr m = Some (m, args, ret, P, Q, s)"*
shows *"wf_stmt Pr (MethodCall y m x) ⟷ length x = length args ∧ length y = length ret ∧ distinct (x @ y)"*
⟨*proof*⟩

lemma *verif_rename_pred:*
assumes *"wf_renaming t"*
shows *"(rename_pred P t) a ⟷ P (rename_state a (rename_inv t))"*
⟨*proof*⟩

lemma *smaller_same_one_side:*
assumes *"wf_renaming t"*
and *"a << b"*
shows *"rename_state a t << rename_state b t"*
⟨*proof*⟩

lemma *smaller_same:*
assumes *"wf_renaming t"*
shows *"a << b ⟷ rename_state a t << rename_state b t"*
⟨*proof*⟩

lemma *rename_pred_same_supported:*
assumes *"supported P"*
and *"wf_renaming t"*
shows *"supported (rename_pred P t)"*
⟨*proof*⟩

lemma *well_defined_assert_monoin:*
assumes *"wf_assert P"*
and *"well_defined_assert P a"*
and *"a << b"*
shows *"well_defined_assert P b"*
⟨*proof*⟩

lemma *rename_pred_same_intui:*
assumes *"wf_assert P"*
and *"wf_renaming t"*
shows *"wf_assert (rename_pred P t)"*
⟨*proof*⟩

lemma *same_wf_rename:*
assumes *"wf_assert P"*
and *"wf_renaming t"*
shows *"wf_assert (rename_pred P t)"*
⟨*proof*⟩

lemma *wf_stmt_wf_renaming:*
assumes *"wf_renaming t"*

```
  shows "wf_stmt Pr s ⟶ wf_stmt Pr (rename s t)"
⟨proof⟩

lemma rename_all:
  assumes "wf_renaming t"
  shows "⋀φ. wf_program Pr ⟹ wf_stmt Pr s ⟹ ver Pr {φ} s ⟹ ver
Pr {rename_state φ t} (rename s t) ∧ sem Pr {rename_state φ t} (rename
s t) = rename_set (sem Pr {φ} s) t"
⟨proof⟩

lemma rename_ver:
  assumes "wf_renaming t"
      and "wf_program Pr"
      and "wf_stmt Pr s"
  shows "ver Pr {φ} s ⟷ ver Pr {rename_state φ t} (rename s t)" (is
"?a ⟷ ?b")
⟨proof⟩

lemma rename_sem:
  assumes "wf_renaming t"
      and "wf_program Pr"
      and "wf_stmt Pr s"
      and "ver Pr {φ} s"
  shows "sem Pr {rename_state φ t} (rename s t) = rename_set (sem Pr
{φ} s) t"
  ⟨proof⟩

lemma rename_ver_set:
  assumes "ver Pr A s"
  and "wf_renaming t"
  and "wf_program Pr"
  and "wf_stmt Pr s"
  shows "ver Pr (rename_set A t) (rename s t)"
  ⟨proof⟩

lemma method_verif_rename:
  assumes "ver Pr {u} (Var (args @ ret) ; Inhale P ; s ; Exhale Q)"
      and "wf_renaming (args @ ret, x @ y, l, do)"
      and "wf_program Pr"
      and "wf_stmt Pr (Var (args @ ret) ; Inhale P ; s ; Exhale Q)"
  shows "ver Pr {u} (Var (x @ y) ; Inhale (rename_pred P (args @ ret,
x @ y, l, do)) ;
      rename s (args @ ret, x @ y, l, do) ; Exhale (rename_pred Q (args
@ ret, x @ y, l, do)))"
⟨proof⟩

lemma sem_method:
  assumes "ver Pr {φ} (MethodCall y m x)"
      and "wf_stmt Pr (MethodCall y m x)"
```

```
  shows "∃ args ret P Q s. get_method Pr m = Some (m, args, ret, P, Q,
s) ∧
        sem Pr {φ} (MethodCall y m x) = sem Pr {φ} (Exhale (rename_pred
P (args @ ret, x @ y, [], [])); Havoc y ;
                      Inhale (rename_pred Q (args @ ret, x @ y, [], [])))"
  ⟨proof⟩

lemma wf_program_method:
  assumes "wf_program Pr"
      and "get_method Pr name = Some m"
    shows "wf_method Pr m"
  ⟨proof⟩

lemma ver_method:
  assumes "ver Pr {φ} (MethodCall y m x)"
      and "wf_stmt Pr (MethodCall y m x)"
      and "wf_program Pr"
  shows "∃args ret P Q s.
          wf_method Pr (m, args, ret, P, Q, s) ∧
          ver Pr {φ} (Exhale (rename_pred P (args @ ret, x @ y, [], []))
; Havoc y ; Inhale (rename_pred Q (args @ ret, x @ y, [], []))) ∧
          (set x ∪ set y) ⊆ σ φ ∧
          length args = length x ∧ length ret = length y"
⟨proof⟩

lemma modif_property_other:
  "wf_program Pr ∧ wf_other Pr other ∧ ver Pr {a} (Other other) ∧ c ∈
sem Pr {a} (Other other) ⟶ σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif (Other
other))"
  ⟨proof⟩

definition modif_property :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt
⇒ bool" where
  "modif_property Pr s ⟷ (∀a c. wf_program Pr ∧ wf_stmt Pr s ∧ ver
Pr {a} s ∧ c ∈ sem Pr {a} s ⟶ σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif
s))"

lemma modif_property_inhale:
  "modif_property Pr (Inhale P)"
  ⟨proof⟩

lemma modif_property_exhale:
  "modif_property Pr (Exhale P)"
  ⟨proof⟩

lemma modif_property_havoc:
  "modif_property Pr (Havoc y)"
⟨proof⟩
```

lemma *store_modif_sem_seq:*
assumes *"s = s1 ; s2"*
and *"⋀a c. wf_program Pr ∧ wf_stmt Pr s1 ∧ ver Pr {a} s1 ∧ c ∈ sem Pr {a} s1 ⟹ σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif s1)"*
and *"⋀a c. wf_program Pr ∧ wf_stmt Pr s2 ∧ ver Pr {a} s2 ∧ c ∈ sem Pr {a} s2 ⟹ σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif s2)"*
and *"wf_program Pr ∧ wf_stmt Pr s ∧ ver Pr {a} s ∧ c ∈ sem Pr {a} s"*
shows *"σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif s)"*
⟨*proof*⟩

lemma *modif_property_seq:*
assumes *"modif_property Pr s1"*
and *"modif_property Pr s2"*
shows *"modif_property Pr (s1 ; s2)"*
⟨*proof*⟩

lemma *modif_property_assume:*
"modif_property Pr (Assume b)"
⟨*proof*⟩

lemma *wf_wf_renaming:*
assumes *"wf_program Pr"*
and *"wf_stmt Pr (MethodCall y m x)"*
and *"get_method Pr m = Some (m, args, ret, P, Q, s)"*
shows *"wf_renaming (args @ ret, x @ y, l, do)"*
and *"wf_renaming (args @ ret, x @ y, l, do)"*
⟨*proof*⟩

lemma *store_modif_sem: "wf_program Pr ∧ wf_stmt Pr s ∧ ver Pr {a} s ∧ c ∈ sem Pr {a} s ⟹ σ a ⊆ σ c ∧ σ c ⊆ σ a ∪ set (modif s)"*
⟨*proof*⟩

lemma *ver_method_set_ret:*
assumes *"ver Pr {φ} (MethodCall y m x)"*
and *"get_method Pr m = Some (m, args, ret, P, Q, s)"*
and *"a ∈ sem Pr {u} (Var (args @ ret) ; Inhale P ; s ; Exhale Q)"*
and *"ver Pr {u} (Var (args @ ret) ; Inhale P ; s ; Exhale Q)"*
and *"wf_program Pr"*
shows *"set ret ⊆ σ a"*
⟨*proof*⟩

lemma *h_comp_only_pure: "C φ = C (h_comp φ l)"*
⟨*proof*⟩

lemma *h_comp_smaller: "h_comp φ x << φ"*
⟨*proof*⟩

lemma *var_sem_empty:*

```
  "sem Pr {u} (Var x) = h x"
  ⟨proof⟩

lemma havoc_concat:
  "h (a @ b) = h a ⊕⊕ h b"
  ⟨proof⟩

lemma havoc_store_bigger:
  assumes "set x ⊆ σ φ"
  shows "{φ} >> {φ} ⊕⊕ h x"
⟨proof⟩

lemma supported_intui_exhale:
  assumes "supported P"
      and "intuitionistic P"
    shows "ver Pr A (Exhale P) ⟷ A >> Inh P"
⟨proof⟩

lemma int_squared_exhale:
  assumes "supported P"
      and "intuitionistic P"
      and "P φ"
      and "Some φ' = Some φ ⊕ Some r"
    shows "sem Pr {φ'} (Exhale P) = (sem Pr {φ} (Exhale P)) ⊕⊕ {r}"
⟨proof⟩

lemma int_squared_inhale:
  assumes "supported P"
      and "Some φ' = Some φ ⊕ Some r"
      and "wf_assert P"
      and "ver Pr {φ} (Inhale P)"
    shows "sem Pr {φ'} (Inhale P) = (sem Pr {φ} (Inhale P)) ⊕⊕ {r}"
  ⟨proof⟩

lemma int_squared_exhale_for_sets:
  assumes "wf_assert P"
      and "∀ φ∈A. P φ"
    shows "sem Pr (A ⊕⊕ {r}) (Exhale P) ⊆ (sem Pr A (Exhale P)) ⊕⊕
{r}" (is "?ae ⊆ ?be")
⟨proof⟩


lemma wf_assert_monoIn:
  assumes "wf_assert P"
  shows "smonoIn Pr (Inhale P)"
    and "smonoIn Pr (Exhale P)"
  ⟨proof⟩

lemma int_squared_inhale_for_sets:
```

```
  assumes "wf_assert P"
      and "ver Pr A (Inhale P)"
    shows "sem Pr (A ⊕⊕ {r}) (Inhale P) = (sem Pr A (Inhale P)) ⊕⊕
{r}"
⟨proof⟩

lemma smono_comp:
  assumes "smono Pr s1"
      and "smono Pr s2"
    shows "smono Pr (Seq s1 s2)"
⟨proof⟩

lemma smono_if:
  assumes "smono Pr s1"
      and "smono Pr s2"
    shows "smono Pr (If s1 s2)"
⟨proof⟩

fun framing :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) stmt ⇒ bool" where
  "framing Pr st = (smono Pr st ∧
  (∀ φ r. ver Pr {φ} st ∧ list.set (modif st) ∩ σ r = {} ⟶ (sem Pr
({φ} ⊕⊕ {r}) st >> (sem Pr {φ} st ⊕⊕ {r}))))"

lemma framing_set:
  "framing Pr s ⟷ smono Pr s ∧ (∀ A r. ((ver Pr A s ∧ list.set (modif
s) ∩ σ r = {}) ⟶ (sem Pr (A ⊕⊕ {r}) s >> (sem Pr A s ⊕⊕ {r}))))"
⟨proof⟩

lemma simple_rename_set_inv:
  assumes "wf_renaming t"
  shows "rename_set (rename_set A t) (rename_inv t) = A"
⟨proof⟩

lemma simple_state_in_set:
  assumes "wf_renaming t"
  shows "rename_state a t ∈ rename_set A t ⟷ a ∈ A" (is "?a ⟷ ?b")
  ⟨proof⟩

lemma simple_renaming_from_set:
  assumes "x ∈ rename_set A t"
  shows "∃ a ∈ A. x = rename_state a t"
  ⟨proof⟩

lemma rename_set_sum:
  assumes "wf_renaming t"
  shows "rename_set (A ⊕⊕ B) t = rename_set A t ⊕⊕ rename_set B t"
⟨proof⟩

lemma rename_set_smaller:
```

```
  assumes "wf_renaming t"
  shows "A >> B ⟷ rename_set A t >> rename_set B t" (is "?a ⟷ ?b")
⟨proof⟩

lemma mono_renamed:
  assumes "smono Pr s"
      and "wf_renaming t"
      and "wf_program Pr"
      and "wf_stmt Pr s"
    shows "smono Pr (rename s t)"
⟨proof⟩

lemma rename_singleton:
  "rename_set {a} t = {rename_state a t}"
  ⟨proof⟩

lemma rename_sem_set:
  assumes "wf_renaming t"
      and "wf_program Pr"
      and "wf_stmt Pr s"
  shows "sem Pr (rename_set A t) s = rename_set (sem Pr A (rename s (rename_inv
t))) t" (is "?a = ?b")
⟨proof⟩

lemma framing_renamed:
  assumes "framing Pr s"
  and "wf_renaming t"
  and "wf_program Pr"
  and "wf_stmt Pr s"
shows "framing Pr (rename s t)"
⟨proof⟩

lemma assume_false_smono:
  "smono Pr (Assume lfalse)"
⟨proof⟩

lemma inhale_smono:
  assumes "wf_assert P"
  shows "smono Pr (Inhale P)"
⟨proof⟩

lemma exhale_elem:
  assumes "ver Pr {a} (Exhale P)"
    shows "sa ∈ sem Pr {a} (Exhale P) ⟷ (∃ i r. Some a = Some i ⊕ Some
r ∧ i ∈ Inh P ∧ Some sa = s_core i ⊕ Some r)" (is "?a ⟷ ?b")
  ⟨proof⟩

lemma exhale_smono:
  assumes "wf_assert P"
```

```
    shows "smono Pr (Exhale P)"
⟨proof⟩

lemma havoc_smono:
  "smono Pr (Havoc x)"
⟨proof⟩

lemma smono_out_equiv:
  assumes "⋀φ. ver Pr {φ} s2 ⟹ sem Pr {φ} s1 = sem Pr {φ} s2"
      and "smonoOut Pr s1"
      and "⋀φ. ver Pr {φ} s2 ⟹ ver Pr {φ} s1"
      and "smonoIn Pr s2"
    shows "smonoOut Pr s2"
  ⟨proof⟩

lemma method_smono_in:
  assumes "wf_program Pr"
      and "wf_stmt Pr (MethodCall y m x)"
    shows "smonoIn Pr (MethodCall y m x)"
⟨proof⟩

lemma method_smono:
  assumes "wf_program Pr"
      and "wf_stmt Pr (MethodCall y m x)"
    shows "smono Pr (MethodCall y m x)" (is "smono Pr ?s")
⟨proof⟩

fun ver_program_aux :: "('a, 'b, 'c) program ⇒ ('a, 'b, 'c) program ⇒
bool" where
  "ver_program_aux Pr [] ⟷ True"
| "ver_program_aux Pr ((_, args, ret, P, Q, s) # q) ⟷ ver Pr {u} (Var
(args @ ret) ; Inhale P ; s ; Exhale Q) ∧ ver_program_aux Pr q"

fun ver_program :: "('a, 'b, 'c) program ⇒ bool" where
  "ver_program Pr ⟷ ver_program_aux Pr Pr"

end

end
```

A.4 Soundness

theory *Soundness*
  imports *Semantics*
begin

context *semantics_algebra*
begin

definition *wfm :: "('a, 'b, 'c) program ⇒ 'a assertion ⇒ bool"* **where**
  "wfm Pr b ⟷ smono Pr (Assume b) ∧ smono Pr (Assume (lnot b))"

lemma *wfm_not_same:*
  "lnot (lnot b) = b"
⟨*proof*⟩

lemma *wfm_not:*
  "wfm Pr b ⟷ wfm Pr (lnot b)"
  ⟨*proof*⟩

lemma *while_smono_in:*
  assumes *"wf_assert I"*
      and *"smono Pr s"*
      and *"wfm Pr b"*
  shows *"smonoIn Pr (While b I s)"*
⟨*proof*⟩

lemma *while_smono:*
  assumes *"wf_assert I"*
      and *"smono Pr s"*
      and *"wfm Pr b"*
  shows *"smono Pr (While b I s)"*
⟨*proof*⟩

lemma *exhale_false_single:*
  "¬ (ver Pr {φ} (Exhale lfalse))"
  ⟨*proof*⟩

fun *inlinable :: "('a, 'b, 'c) stmt ⇒ bool"* **where**
  "inlinable (MethodCall _ _ _) = True"
| "inlinable (While _ _ _) = True"
| "inlinable (Seq a b) = (inlinable a ∨ inlinable b)"
| "inlinable (If a b) = (inlinable a ∨ inlinable b)"
| "inlinable _ = False"

fun *inline :: "('a, 'b, 'c) program ⇒ nat => 'b list ⇒ ('a, 'b, 'c) stmt ⇒ ('a, 'b, 'c) stmt"* **where**
  "inline Pr 0 _ (MethodCall _ _ _) = Assume lfalse"
| "inline Pr 0 _ (While b _ _) = Assume (lnot b)"
| "inline Pr (Suc n) l (MethodCall y name x) = (let r = get_method Pr

```
name in
    if r = None then
      (Exhale lfalse)
    else
      let (_, args, ret, _, _, s) = the r in
      let new_s = rename s (args @ ret, x @ y, l, read s) in
      let new_l = l @ read new_s in
      inline Pr n new_l new_s)"
| "inline Pr (Suc n) l (While b I s) = If (Assume b ; inline Pr n l s
; inline Pr n l (While b I s)) (Assume (lnot b))"
| "inline Pr n l (Seq a b) = Seq (inline Pr n l a) (inline Pr n l b)"
| "inline Pr n l (If a b) = If (inline Pr n l a) (inline Pr n l b)"
| "inline Pr 0 _ s = s"
| "inline Pr _ _ Skip = Skip"
| "inline Pr _ _ s = s"

lemma not_inlinable_id:
  "¬ inlinable s ⟹ inline Pr n l s = s"
  ⟨proof⟩

lemma empty_set_goes_empty:
  assumes "sem Pr A s1 = {}"
      and "ver Pr A s1"
    shows "sem Pr A (s1 ; s2) = {}"
⟨proof⟩

lemma semantics_union:
  "(⋃a∈A. sem Pr (f a) s) = sem Pr (⋃a∈A. f a) s"
  ⟨proof⟩

lemma rename_set_union:
  "(⋃a∈A. rename_set (f a) t) = rename_set (⋃a∈A. f a) t"
  ⟨proof⟩

fun
  SC :: "('a, 'b, 'c) program ⇒ nat ⇒ 'b list ⇒ ('a, 'b, 'c) stmt ⇒
bool" and
  inlinable_SC :: "('a, 'b, 'c) program ⇒ nat ⇒ 'b list ⇒ ('a, 'b,
'c) stmt ⇒ bool"
where
  "SC Pr n l st = (if (¬ inlinable st) then smono Pr st else inlinable_SC
Pr n l st)"
| "inlinable_SC Pr (Suc n) l (MethodCall y m x) = (let (_, args, ret,
_, _, s) = the (get_method Pr m) in
    let new_s = rename s (args @ ret, x @ y, l, read s) in let new_l =
l @ read new_s in
    (framing Pr (inline Pr n new_l new_s) ∨ framing Pr s) ∧ SC Pr n new_l
new_s)"
| "inlinable_SC Pr (Suc n) l (While b I s) ⟷ wfm Pr b ∧ (framing Pr
```

```
(inline Pr n l s) ∨ framing Pr s) ∧ SC Pr n l s ∧ SC Pr n l (While b
I s)"
| "inlinable_SC Pr n l (Seq a b) = (SC Pr n l a ∧ SC Pr n l b)"
| "inlinable_SC Pr n l (If a b) = (SC Pr n l a ∧ SC Pr n l b)"
| "inlinable_SC Pr 0 l (While b _ s) ⟷ wfm Pr b ∧ smono Pr s"
| "inlinable_SC Pr 0 _ _ = True"
| "inlinable_SC Pr _ _ _ = undefined"

lemma sc_implies_smono:
  "wf_program Pr ∧ wf_stmt Pr s ∧ SC Pr n l s ⟹ smono Pr s"
  "wf_program Pr ∧ wf_stmt Pr s ∧ inlinable s ∧ inlinable_SC Pr n l s
⟹ smono Pr s"
⟨proof⟩

lemma inlinable_same:
  "inlinable s ⟷ inlinable (rename s t)"
  ⟨proof⟩

lemma not_inlinable_sc:
  assumes "¬ inlinable s"
      and "SC Pr n l s"
      and "wf_renaming t"
      and "wf_program Pr"
      and "wf_stmt Pr s"
    shows "SC Pr n l (rename s t)"
⟨proof⟩

lemma wfm_same:
  assumes "wfm Pr b"
      and "wf_renaming t"
      and "wf_program Pr"
      and "wf_stmt Pr s"
    shows "wfm Pr (rename_pred b t)"
⟨proof⟩

lemma sc_method_implies_new:
  assumes "new_s = rename s (args @ ret, x @ y, l, read s)"
      and "new_l = l @ read new_s"
      and "SC Pr (Suc n) l (MethodCall y m x)"
      and "get_method Pr m = Some (m, args, ret, P, Q, s)"
    shows "(framing Pr (inline Pr n new_l new_s) ∨ framing Pr s) ∧ SC
Pr n new_l new_s"
⟨proof⟩

lemma smono_inline:
  "SC Pr n l s ⟹ smono Pr (inline Pr n l s)"
⟨proof⟩

definition no_inter_single :: "'a ⇒ 'b list ⇒ bool" where
```

```
  "no_inter_single x l ⟷ σ x ⊆ set l"

definition no_inter :: "'a set ⇒ 'b list ⇒ bool" where
  "no_inter A l ⟷ (∀a ∈ A. no_inter_single a l)"

lemma inter_sum_sets:
  assumes "no_inter A l"
      and "no_inter B l"
    shows "no_inter (A ⊕⊕ B) l"
  ⟨proof⟩

definition SP :: "('a, 'b, 'c) program ⇒ nat ⇒ 'b list ⇒ ('a, 'b, 'c)
stmt ⇒ bool" where
  "SP Pr n l s ⟷ (∀A A'. ver_program Pr ∧ wf_program Pr ∧ wf_stmt
Pr s ∧ A' >> A ∧ SC Pr n l s ∧ ver Pr A s ∧ no_inter A l ∧ no_inter
A' l ∧ set (modif s) ⊆ set l ⟶ ver Pr A' (inline Pr n l s) ∧ sem Pr
A' (inline Pr n l s) >> sem Pr A s)"

lemma havoc_int_squared_single:
  assumes "Some a = Some v ⊕ Some r"
      and "set x ∩ σ r = {}"
      and "ver Pr {v} (Havoc x)"
    shows "sem Pr {a} (Havoc x) = sem Pr {v} (Havoc x) ⊕⊕ {r}" (is "?a
= ?b")
⟨proof⟩

lemma havoc_invertible:
  assumes "set V ⊆ σ a"
      and "Some b = Some (h_comp a V) ⊕ Some hv"
      and "hv ∈ h V"
    shows "∃hv' ∈ h V. Some a = Some (h_comp b V) ⊕ Some hv'"
⟨proof⟩

lemma decompo_sigma:
  assumes "σ p = a ∪ b"
      and "a ∩ b = {}"
      and "pure p"
    shows "∃pa pb. Some p = Some pa ⊕ Some pb ∧ σ pa = a ∧ σ pb = b"
⟨proof⟩

lemma smaller_defined:
  assumes "a << b"
      and "b ## r"
    shows "a ## r"
  ⟨proof⟩

lemma pure_defined_sum:
  assumes "Some x = Some xx ⊕ Some p"
      and "pure p"
```

```
      and "xx ## r"
      and "p ## r"
    shows "x ## r"
⟨proof⟩

lemma havoc_int_squared:
  assumes "set x ∩ σ r = {}"
      and "ver Pr A (Havoc x)"
      and "⋀φ. φ ∈ A ⟹ set x ⊆ σ φ"
    shows "sem Pr (A ⊕⊕ {r}) (Havoc x) = sem Pr A (Havoc x) ⊕⊕ {r}"
(is "?a = ?b")
⟨proof⟩

lemma simple_sem_exhale:
  assumes "wf_assert P"
      and "Some φ = Some i ⊕ Some r"
      and "i ∈ Inh P"
      and "Some φ' = s_core i ⊕ Some r"
    shows "sem Pr {φ} (Exhale P) = {φ'}" (is "?a = ?b")
⟨proof⟩

lemma h_set_pure:
  assumes "A >> B"
      and "A >> h x"
    shows "A >> B ⊕⊕ h x"
⟨proof⟩

lemma bigger_h_single:
  assumes "set x ⊆ σ a"
  shows "{a} >> {h_comp a x} ⊕⊕ h x"
⟨proof⟩

lemma core_i_phi_exhale:
  assumes "Some i_phi = Some i ⊕ Some x"
      and "Some core_i_phi = s_core i ⊕ Some x"
      and "i ∈ Inh P"
      and "supported P"
      and "P i_phi"
    shows "sem Pr {i_phi} (Exhale P) = {core_i_phi}" (is "?a = ?b")
⟨proof⟩



lemma get_method_inlined:
  assumes "get_method Pr m = Some (m, args, ret, P, Q, s)"
      and "new_s = rename s (args @ ret, x @ y, l, read s)"
      and "new_l = l @ read new_s"
    shows "inline Pr (Suc n) l (MethodCall y m x) = inline Pr n new_l
new_s"
```

⟨*proof*⟩

lemma *no_inter_inline_general:*
"wf_program Pr ∧ wf_stmt Pr s ⟹ set l ∩ set (modif (inline Pr n l s)) ⊆ set (modif s)"
⟨*proof*⟩

lemma *ver_program_method_verif_aux:*
"ver_program_aux Pr Pr_bis ∧ Some (m, args, ret, P, Q, s) = get_method Pr_bis m ⟹ ver Pr {u} (Var (args @ ret) ; Inhale P ; s ; Exhale Q)"
⟨*proof*⟩

lemma *ver_program_method_verif:*
assumes *"ver_program Pr"*
and *"Some (m, args, ret, P, Q, s) = get_method Pr m"*
shows *"ver Pr {u} (Var (args @ ret) ; Inhale P ; s ; Exhale Q)"*
⟨*proof*⟩

lemma *verif_inhale_var_alone:*
assumes *"ver Pr {u} (Var (args @ ret) ; Inhale P)"*
shows *"sem Pr {u} (Var (args @ ret) ; Inhale P) = h args ⊕⊕ h ret ⊕⊕ Inh P"*
⟨*proof*⟩

lemma *verif_inhale_var:*
assumes *"ver Pr {u} (Var (args @ ret) ; Inhale P ; s)"*
shows *"sem Pr {u} (Var (args @ ret) ; Inhale P ; s) = sem Pr (h args ⊕⊕ h ret ⊕⊕ Inh P) s"*
⟨*proof*⟩

lemma *ver_asso:*
"ver Pr A (s1 ; (s2 ; s3)) ⟷ ver Pr A ((s1 ; s2) ; s3)"
⟨*proof*⟩

lemma *sem_asso:*
assumes *"ver Pr A (s1 ; s2 ; s3)"*
shows *"sem Pr A (s1 ; (s2 ; s3)) = sem Pr A ((s1 ; s2) ; s3)"*
⟨*proof*⟩

A.4.1 Method case

lemma *method_inlining_induct:*
assumes *"SP Pr n new_l new_s"*
and *"Some (m, args, ret, P, Q, s) = get_method Pr m"*
and *"new_s = rename s (args @ ret, x @ y, l, read s)"*
and *"new_l = l @ read new_s"*
shows *"SP Pr (Suc n) l (MethodCall y m x)"*
⟨*proof*⟩

lemma *bigger_set_singleton:*
assumes *"⋀a. a ∈ A ⟹ (∃ b ∈ B. sem Pr {a} s1 >> sem Pr {b} s2)"*
shows *"sem Pr A s1 >> sem Pr B s2"*
⟨*proof*⟩

lemma *instantiate_SP:*
assumes *"ver_program Pr ∧ wf_program Pr ∧ wf_stmt Pr s ∧ {φ'} >> {φ} ∧ SC Pr n l s ∧ ver Pr {φ} s ∧ no_inter {φ'} l ∧ no_inter {φ} l ∧ set (modif s) ⊆ set l"*
and *"SP Pr n l s"*
shows *"ver Pr {φ'} (inline Pr n l s) ∧ sem Pr {φ'} (inline Pr n l s) >> sem Pr {φ} s"*
⟨*proof*⟩

definition *f :: "'a set ⇒ 'a assertion ⇒ 'a set"* **where**
"f A b = Set.filter b A"

lemma *f_singleton:*
"(⋃a∈A. f {a} b) = f A b" ⟨*proof*⟩

lemma *sem_assume_filter:*
assumes *"ver Pr A (Assume b)"*
shows *"sem Pr A (Assume b) = f A b"*
⟨*proof*⟩

lemma *f_inclusion:*
"f A b ⊆ A" ⟨*proof*⟩

lemma *wfm_f:*
assumes *"wfm Pr b"*
and *"A >> B"*
and *"ver Pr B (Assume b)"*
shows *"f A b >> f B b"*
⟨*proof*⟩

lemma *if_assume_then_true:*
assumes *"x ∈ sem Pr A (Assume b)"*
and *"ver Pr A (Assume b)"*
shows *"b x"*
⟨*proof*⟩

lemma *after_while:*
assumes *"ver Pr A (While b I s)"*
and *"sa ∈ sem Pr A (While b I s)"*
shows *"lnot b sa"*
⟨*proof*⟩

lemma *general_same_f:*
assumes *"⋀a. a ∈ A ⟹ b a"*

shows *"f A b = A"*
⟨*proof*⟩

lemma *after_while_same_f:*
assumes *"ver Pr A (While b I s)"*
shows *"f (sem Pr A (While b I s)) (lnot b) = sem Pr A (While b I s)"*
⟨*proof*⟩

lemma *after_inline_while:*
"ver Pr A (inline Pr n l (While b I s)) ⟹ sa ∈ sem Pr A (inline Pr n l (While b I s)) ⟹ (lnot b sa)"
⟨*proof*⟩

lemma *after_inline_while_same_f:*
assumes *"ver Pr A (inline Pr n l (While b I s))"*
shows *"f (sem Pr A (inline Pr n l (While b I s))) (lnot b) = sem Pr A (inline Pr n l (While b I s))"*
⟨*proof*⟩

lemma *wfm_bigger:*
assumes *"c << a"*
and *"wfm Pr b"*
and *"b c"*
and *"ver Pr {c} (Assume b)"*
shows *"b a"*
⟨*proof*⟩

lemma *simple_SP:*
assumes *"⋀A A'. ver_program Pr ∧ wf_program Pr ∧ wf_stmt Pr s ∧ A' >> A ∧ SC Pr n l s ∧ ver Pr A s ∧ no_inter A' l ∧ no_inter A l ∧ set (modif s) ⊆ set l ⟹ ver Pr A' (inline Pr n l s) ∧ sem Pr A' (inline Pr n l s) >> sem Pr A s"*
shows *"SP Pr n l s"*
⟨*proof*⟩

lemma *assume_false_sem:*
"sem Pr A (Assume lfalse) = {}" ⟨*proof*⟩

lemma *verif_exhale_exists_decompo:*
assumes *"ver Pr {φ} (Exhale I)"*
and *"wf_assert I"*
shows *"∃i r. i ∈ Inh I ∧ Some φ = Some i ⊕ Some r"*
⟨*proof*⟩

lemma *assume_not_bigger:*
assumes *"ver Pr {φ} (While b I s)"*
and *"wf_assert I"*
shows *"{φ} >> sem Pr {φ} (Exhale I ; Havoc (modif s) ; Inhale I)"*
⟨*proof*⟩

lemma *assume_not_bigger_real:*
assumes *"ver Pr A (While b I s)"*
and *"wf_assert I"*
shows *"A >> sem Pr A (Exhale I ; Havoc (modif s) ; Inhale I)"*
⟨*proof*⟩

lemma *add_h_comp_core:*
assumes *"Some x = Some a ⊕ Some b"*
and *"set V ⊆ σ x"*
shows *"Some x = Some a ⊕ Some (h_comp b V) ⊕ s_core x"*
⟨*proof*⟩

lemma *h_comp_set_add_two:*
assumes *"Some x = Some a ⊕ Some b"*
shows *"{h_comp x V} = {h_comp a V} ⊕⊕ {h_comp b V}"*
⟨*proof*⟩

lemma *h_comp_set_add:*
assumes *"Some x = Some a ⊕ Some b ⊕ Some c ⊕ Some d ⊕ Some e"*
shows *"{h_comp x V} = {h_comp a V} ⊕⊕ {h_comp b V} ⊕⊕ {h_comp c V} ⊕⊕{h_comp d V} ⊕⊕ {h_comp e V}"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *h_comp_sum_set_smaller_pure:*
assumes *"pure a"*
and *"a << b"*
shows *"{h_comp a V} ⊕⊕ {h_comp b V} = {h_comp b V}"*
⟨*proof*⟩

lemma *h_comp_smaller_elem:*
assumes *"h_comp |a| V << x"*
shows *"{h_comp |a| V} ⊕⊕ {h_comp |x| V} = {h_comp |x| V}"*
⟨*proof*⟩

lemma *core_four:*
assumes *"Some a = Some b ⊕ Some c ⊕ Some d ⊕ Some e"*
shows *"s_core a = s_core b ⊕ s_core c ⊕ s_core d ⊕ s_core e"*
⟨*proof*⟩

lemma *h_comp_set_add_four:*
assumes *"Some x = Some a ⊕ Some b ⊕ Some c ⊕ Some d"*
shows *"{h_comp x V} = {h_comp a V} ⊕⊕ {h_comp b V} ⊕⊕ {h_comp c V} ⊕⊕{h_comp d V}"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *framing_set_singleton:*
assumes *"σ r ∩ set (modif s) = {}"*
and *"ver Pr A s"*

```
      and "framing Pr s"
    shows "sem Pr (A ⊕⊕ {r}) s >> sem Pr A s ⊕⊕ {r}"
⟨proof⟩

lemma sigma_sem_havoc:
  assumes "ver Pr {a} (Havoc V)"
    and "x ∈ sem Pr {a} (Havoc V)"
  shows "σ x = σ a ∪ set V"
⟨proof⟩

lemma bigger_and_ver_assume:
  assumes "ver Pr B (Assume b)"
      and "wfm Pr b"
      and "A >> B"
    shows "f A b >> f B b"
      and "f A (lnot b) >> f B (lnot b)"
  ⟨proof⟩

lemma wfm_f_and_sum:
  assumes "wfm Pr b"
      and "ver Pr A (Assume b)"
    shows "f (A ⊕⊕ D) b = f A b ⊕⊕ D" (is "?a = ?b")
⟨proof⟩

lemma not_empty_exhale:
  assumes "ver Pr {φ} (Exhale P)"
      and "wf_assert P"
  shows "sem Pr {φ} (Exhale P) ≠ {}"
  ⟨proof⟩

lemma not_empty_havoc:
  assumes "ver Pr {φ} (Havoc V)"
  shows "sem Pr {φ} (Havoc V) ≠ {}"
⟨proof⟩

lemma not_empty_comp:
  assumes "⋀a. sem Pr {a} s1 ≠ {}"
    and "⋀b. sem Pr {b} s2 ≠ {}"
    and "ver Pr {c} (s1; s2)"
  shows "sem Pr {c} (s1 ; s2) ≠ {}"
⟨proof⟩

lemma h_comp_add_single:
  assumes "{a} = {b} ⊕⊕ {c}"
  shows "{h_comp a x} = {h_comp b x} ⊕⊕ {h_comp c x}"
  ⟨proof⟩
```

A.4.2 Loop case

lemma *loop_inlining_induct:*
"SP Pr n l (While b I s) ∧ SP Pr n l s ⟹ SP Pr (Suc n) l (While b I s)"
⟨*proof*⟩

lemma *sem_loop_set:*
assumes *"ver Pr A (While b I s)"*
shows *"sem Pr A (While b I s) = sem Pr A (Exhale I ; Havoc (modif s) ; Inhale I ; Assume (lnot b))"* (**is** *"?a = ?b"*)
⟨*proof*⟩

lemma *wf_none_impossible:*
assumes *"wf_stmt Pr (MethodCall y m x)"*
and *"wf_program s"*
shows *"get_method Pr m ≠ None"*
⟨*proof*⟩

A.4.3 Soundness proof

lemma *soundness_invariant:*
"SP Pr n l s"
⟨*proof*⟩

theorem *soundness:*
assumes *"wf_program Pr"*
and *"wf_stmt Pr s"*
and *"SC Pr n (modif s) s"*
and *"ver_program Pr"*
and *"ver Pr {u} s"*
shows *"ver Pr {u} (inline Pr n (modif s) s)"*
⟨*proof*⟩

end

end

A.5 Renaming

```
theory Renaming
  imports Main
begin

fun sum :: "nat list ⇒ nat" where
  "sum [] = 1"
| "sum (t # q) = t + (sum q)"

lemma member_smaller:
  "x ∈ set l ⟹ x ≤ sum l + 1"
  ⟨proof⟩

type_synonym rename_t = "nat list × nat list × nat list × nat list"

fun wf_renaming :: "rename_t ⇒ bool" where
  "wf_renaming (l1, l2, l3, l4) ⟷ (length l1 = length l2 ∧ distinct
l1 ∧ distinct l2)"

fun get_value :: "nat ⇒ nat list ⇒ nat list ⇒ nat option" where
  "get_value x (ta # qa) (tb # qb) = (if x = ta then Some tb else get_value
x qa qb)"
| "get_value x _ _ = None"

lemma not_in_list_none:
  "x ∉ set a ⟹ get_value x a b = None"
  ⟨proof⟩

lemma elem_get_value:
  "i < length a ∧ distinct a ∧ distinct b ∧ length a = length b ⟹ get_value
(a ! i) a b = Some (b ! i)"
⟨proof⟩

definition sum_all :: "nat list ⇒ nat list ⇒ nat list ⇒ nat list ⇒ nat"
where
  "sum_all a b c d = sum a + sum b + sum c + sum d"

lemma sum_greater_one:
  "sum l ≥ 1"
  ⟨proof⟩

lemma sum_all_commut:
  "sum_all a b c d = sum_all b c d a"
  ⟨proof⟩

lemma x_greater_not_in:
  assumes "x ≥ sum_all a b c d"
  shows "x ∉ set a"
⟨proof⟩
```

lemma *x_greater_not_in_all:*
  assumes *"x ≥ sum_all a b c d"*
  shows *"x ∉ set a ∪ set b ∪ set c ∪ set d"*
⟨*proof*⟩

definition *injective :: "(nat ⇒ nat) ⇒ bool"* **where**
  "injective f ⟷ (∀a b. f a = f b ⟶ a = b)"

definition *surjective :: "(nat ⇒ nat) ⇒ bool"* **where**
  "surjective f ⟷ (∀y. ∃x. f x = y)"

definition *bijective :: "(nat ⇒ nat) ⇒ bool"* **where**
  "bijective f ⟷ injective f ∧ surjective f"

definition *id_above :: "(nat ⇒ nat) ⇒ bool"* **where**
  "id_above f ⟷ (∃m. ∀x ≥ m. f x = x)"

definition *same_lists :: "(nat ⇒ nat) ⇒ nat list ⇒ nat list ⇒ bool"*
where
  "same_lists f a b ⟷ (∀i < length a. f (a ! i) = b ! i)"

lemma *easy_append_same_lists:*
  assumes *"same_lists f a b"*
      and *"f x = y"*
    shows *"same_lists f (x # a) (y # b)"* (**is** *"same_lists f ?a ?b"*)
⟨*proof*⟩

lemma *induction_create:*
  "length a = length b ∧ distinct a ∧ distinct b ⟹ (∃f. bijective
f ∧ id_above f ∧ same_lists f a b)"
⟨*proof*⟩

fun *get_f :: "nat list ⇒ nat list ⇒ (nat ⇒ nat)"* **where**
  "get_f a b = (SOME f. bijective f ∧ id_above f ∧ same_lists f a b)"

lemma *get_f_works_well:*
  assumes *"length a = length b"*
      and *"distinct a"*
      and *"distinct b"*
      and *"f = get_f a b"*
    shows *"bijective f ∧ id_above f ∧ same_lists f a b"*
  ⟨*proof*⟩

fun *first_rename :: "rename_t ⇒ nat ⇒ nat"* **where**
  "first_rename (a, b, c, d) x = (let r = get_value x a b in
  if r = None then x + sum_all a b c d
  else the r)"

```
lemma first_rename_in_a:
  assumes "wf_renaming (a, b, c, d)"
      and "i < length a"
    shows "first_rename (a, b, c, d) (a ! i) = b ! i"
  ⟨proof⟩

lemma first_rename_in_d:
  assumes "wf_renaming (a, b, c, d)"
      and "x ∈ set d"
      and "x ∉ set a"
    shows "first_rename (a, b, c, d) x = x + sum_all a b c d"
  ⟨proof⟩

fun distinctify :: "nat list ⇒ nat list" where
  "distinctify [] = []"
| "distinctify (t # q) = (if t ∈ set q then distinctify q else t # distinctify
q)"

lemma same_elems_distinctify:
  "set l = set (distinctify l)"
  ⟨proof⟩

lemma distinctify_is_distinct:
  "distinct (distinctify l)"
  ⟨proof⟩

lemma f_rename_b:
  assumes "wf_renaming (a, b, c, d)"
      and "domain = distinctify (a @ d)"
      and "images = map (first_rename (a, b, c, d)) domain"
      and "f = get_f domain images"
    shows "bijective f ∧ id_above f ∧ same_lists f domain images"
⟨proof⟩

fun rename_b :: "nat ⇒ rename_t ⇒ nat" where
  "rename_b x (a, b, c, d) = (let domain = distinctify (a @ d) in
  get_f domain (map (first_rename (a, b, c, d)) domain) x)"

lemma rename_b_in_a:
  assumes "wf_renaming (a, b, c, d)"
      and "i < length a"
    shows "rename_b (a ! i) (a, b, c, d) = b ! i"
⟨proof⟩

lemma rename_b_in_d:
  assumes "wf_renaming (a, b, c, d)"
      and "x ∈ set d - set a"
    shows "rename_b x (a, b, c, d) = x + sum_all a b c d"
⟨proof⟩
```

```
fun gen_list :: "nat ⇒ nat list" where
  "gen_list 0 = []"
  | "gen_list (Suc n) = gen_list n @ [n]"

lemma gen_list_length:
  "length (gen_list n) = n"
  ⟨proof⟩

lemma gen_list_def:
  "⋀i. i < n ⟹ gen_list n ! i = i"
  ⟨proof⟩

lemma gen_list_member:
  "x ∈ set (gen_list n) ⟷ x < n"
  ⟨proof⟩

lemma distinct_gen_list:
  "distinct (gen_list n)"
  ⟨proof⟩

fun invert_f :: "(nat ⇒ nat) ⇒ (nat ⇒ nat)" where
  "invert_f f = (λx. THE y. f y = x)"

lemma the_y_works:
  assumes "bijective f"
  shows "f (THE y. f y = x) = x"
⟨proof⟩

lemma bijective_then_works:
  assumes "bijective f"
  shows "invert_f f (f x) = x"
  ⟨proof⟩

lemma bijective_inverse_other_dir:
  assumes "bijective f"
  shows "f (invert_f f x) = x"
  ⟨proof⟩

lemma bijective_inv:
  assumes "bijective f"
  shows "bijective (invert_f f)"
  ⟨proof⟩

fun rename_inv_b :: "rename_t ⇒ rename_t" where
  "rename_inv_b (a, b, c, d) = (let domain = distinctify (a @ d) in
  let f = get_f domain (map (first_rename (a, b, c, d)) domain) in
  let m = (SOME m. ∀x ≥ m. f x = x) in
  let ante = gen_list m in
```

```
  (ante, map (invert_f f) ante, [], ante))"

lemma wf_renaming_rename_inv_b:
  assumes "wf_renaming (a, b, c, d)"
  shows "wf_renaming (rename_inv_b (a, b, c, d))"
⟨proof⟩

fun rename_a :: "nat ⇒ rename_t ⇒ nat" where
  "rename_a x (a, b, c, d) = (let r = get_value x a b in
  if r = None then x else the r)"

fun rename_elem :: "nat ⇒ rename_t ⇒ nat" where
  "rename_elem x (a, b, c, d) = (if c = [] ∧ set a = set b then rename_a
x (a, b, c, d)
  else rename_b x (a, b, c, d))"

lemma gen_list_set:
  "set (gen_list m) = {i. i < m}"
⟨proof⟩

lemma rename_elem_rename_inv_b:
  assumes "wf_renaming t"
  shows "rename_elem x (rename_inv_b t) = rename_a x (rename_inv_b t)"
⟨proof⟩

lemma id_above_then_m_works:
  assumes "id_above f"
      and "m = (SOME m. ∀x ≥ m. f x = x)"
    shows "∀x ≥ m. f x = x"
⟨proof⟩

lemma rename_inv_b_is_invert_f:
  assumes "wf_renaming (a, b, c, d)"
      and "domain = distinctify (a @ d)"
      and "images = map (first_rename (a, b, c, d)) domain"
      and "f = get_f domain images"
    shows "rename_a x (rename_inv_b (a, b, c, d)) = invert_f f x"
⟨proof⟩

lemma rename_inv_b_is_inv:
  assumes "wf_renaming (a, b, c, d)"
  shows "rename_elem (rename_b x (a, b, c, d)) (rename_inv_b (a, b, c,
d)) = x"
⟨proof⟩

fun rename_inv_a :: "rename_t ⇒ rename_t" where
  "rename_inv_a (a, b, c, d) = (b, a, [], map (λelem. rename_elem elem
(a, b, c, d)) d)"
```

```
lemma wf_rename_inv_b:
  assumes "wf_renaming (a, b, c, d)"
  shows "wf_renaming (rename_inv_a (a, b, c, d))"
  ⟨proof⟩

fun rename_inv :: "rename_t ⇒ rename_t" where
  "rename_inv (a, b, c, d) = (if c = [] ∧ set a = set b then rename_inv_a
(a, b, c, d)
  else rename_inv_b (a, b, c, d))"

lemma wf_rename_inv:
  assumes "wf_renaming t"
  shows "wf_renaming (rename_inv t)"
  ⟨proof⟩

lemma rename_inv_a_is_inv:
  assumes "wf_renaming (a, b, [], d)" (is "wf_renaming ?t")
      and "set a = set b"
  shows "rename_elem (rename_a x ?t) (rename_inv_a ?t) = x"
⟨proof⟩

lemma rename_removes_vars_b:
  assumes "wf_renaming (a, b, c, d)"
      and "var ∈ set d"
      and "rename_b var (a, b, c, d) ∈ set a ∪ set c"
    shows "rename_b var (a, b, c, d) ∈ set b"
⟨proof⟩

lemma rename_inv_works:
  assumes "wf_renaming t"
  shows "rename_elem (rename_elem x t) (rename_inv t) = x"
  ⟨proof⟩

lemma rename_removes_vars:
  assumes "wf_renaming (a, b, c, d)"
      and "var ∈ set d"
      and "rename_elem var (a, b, c, d) ∈ set a ∪ set c"
    shows "rename_elem var (a, b, c, d) ∈ set b"
⟨proof⟩

lemma rename_elem_in_a_b:
  assumes "wf_renaming (a, b, c, d)"
      and "i < length a"
    shows "rename_elem (a ! i) (a, b, c, d) = b ! i"
⟨proof⟩

end
```

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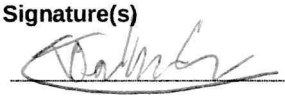
MIONS (FRANCE), 07.04.2020	

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