Fractional Resources in Unbounded Separation Logic

Thibault Dardinier, Peter Müller, and Alexander J. Summers

Separation logic

```
method caller() {

  a := new ObjectF(5)

  b := new ObjectF(7)

  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

```
method callee(b: Ref)


{
  ... // reads b.f
}
```

Separation logic

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}
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a.f

Full (exclusive) permission
Permits to read **and write** *a.f*

$a.f \mapsto \_$

```
method callee(b: Ref)

{
  ... // reads b.f
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Separation logic

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method caller() {

  a := new ObjectF(5) // a.f = 5
  b := new ObjectF(7) // b.f = 7
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
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$a.f \mapsto \_$

Full (exclusive) permission
Permits to read **and write** *a.f*

a.f

a.f b.f

```
method callee(b: Ref)


{
  ... // reads b.f
}
```

Separation logic

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f]
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

a.f ↦ _

Full (exclusive) permission
Permits to read **and write** *a.f*

```
method callee(b: Ref)
  requires [b.f]
  ensures  [b.f]
{
  ... // reads b.f
}
```

Separation logic

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f]
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

$a.f \mapsto \_$

Full (exclusive) permission
Permits to read **and write** *a.f*

$$\frac{\{P\}\ C\ \{Q\} \qquad mod(C) \cap fv(R) = \emptyset}{\{P * R\}\ C\ \{Q * R\}}\ (Frame)$$

```
method callee(b: Ref)
  requires [b.f]
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Separation logic

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f]
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

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Permits to read **and write** *a.f*

$$\text{a.f} \mapsto \_$$

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```
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  requires [b.f]
  ensures  [b.f]
{
  ... // reads b.f
}
```

Separation logic

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f]
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

a.f $\mapsto$ _

Full (exclusive) permission
Permits to read **and write** *a.f*

$$\frac{\{P\}\ C\ \{Q\} \qquad mod(C) \cap fv(R) = \emptyset}{\{P * R\}\ C\ \{Q * R\}}\ (Frame)$$

Disjoint permissions

```
method callee(b: Ref)
  requires [b.f]
  ensures [b.f]
{
  ... // reads b.f
}
```

Separation logic

```
method caller() {

  a := new ObjectF(5) // a.f = 5
```

a.f

a.f ↦ _

Full (exclusive) permission
Permits to read **and write** *a.f*

```
  b := new ObjectF(7) // b.f = 7
```

a.f b.f

```
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

$$\frac{\{P\}\ C\ \{Q\} \qquad mod(C) \cap fv(R) = \emptyset}{\{P * R\}\ C\ \{Q * R\}}\ (\mathit{Frame})$$

Disjoint permissions

```
method callee(b: Ref)
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Separation logic

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  a := new ObjectF(5) // a.f = 5
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  callee(b)
  assert a.f == 5
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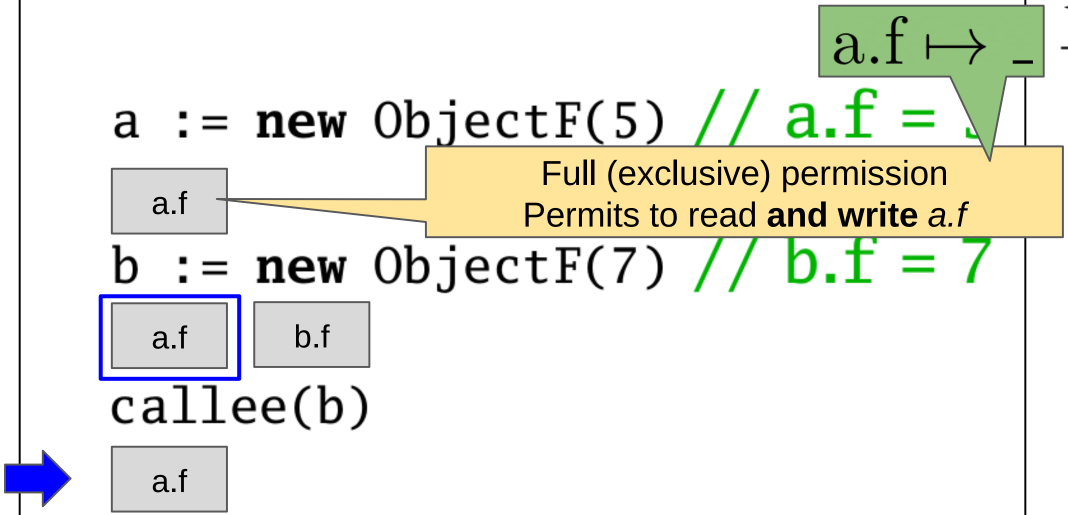
}
```



$$\frac{\{P\}\ C\ \{Q\} \qquad mod(C) \cap fv(R) = \varnothing}{\{P * R\}\ C\ \{Q * R\}}\ (Frame)$$

Disjoint permissions

```
method callee(b: Ref)
  requires b.f
  ensures b.f
{
  ... // reads b.f
}
```

Separation logic

```
method caller() {

   a := new ObjectF(5) // a.f = 5
   [a.f]
   b := new ObjectF(7) // b.f = 7
   [a.f] [b.f]
   callee(b)
   [a.f]
   assert a.f == 5
   assert b.f == 7
}
```

$a.f \mapsto \_$

Full (exclusive) permission
Permits to read **and write** *a.f*

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```
method callee(b: Ref)
   requires [b.f]
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{
   ... // reads b.f
}
```

Separation logic

```
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  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f]
  callee(b)
  [a.f] [b.f]
  assert a.f == 5
  assert b.f == 7
}
```

$a.f \mapsto \_$

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Permits to read **and write** *a.f*

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Disjoint permissions

```
method callee(b: Ref)
  requires [b.f]
  ensures [b.f]
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  ... // reads b.f
}
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Separation logic

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f]
  callee(b)
  [a.f] [b.f]
  assert a.f == 5
  assert b.f == 7
}
```

$a.f \mapsto \_$

Full (exclusive) permission
Permits to read **and write** *a.f*

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Disjoint permissions

```
method callee(b: Ref)
  requires [b.f]
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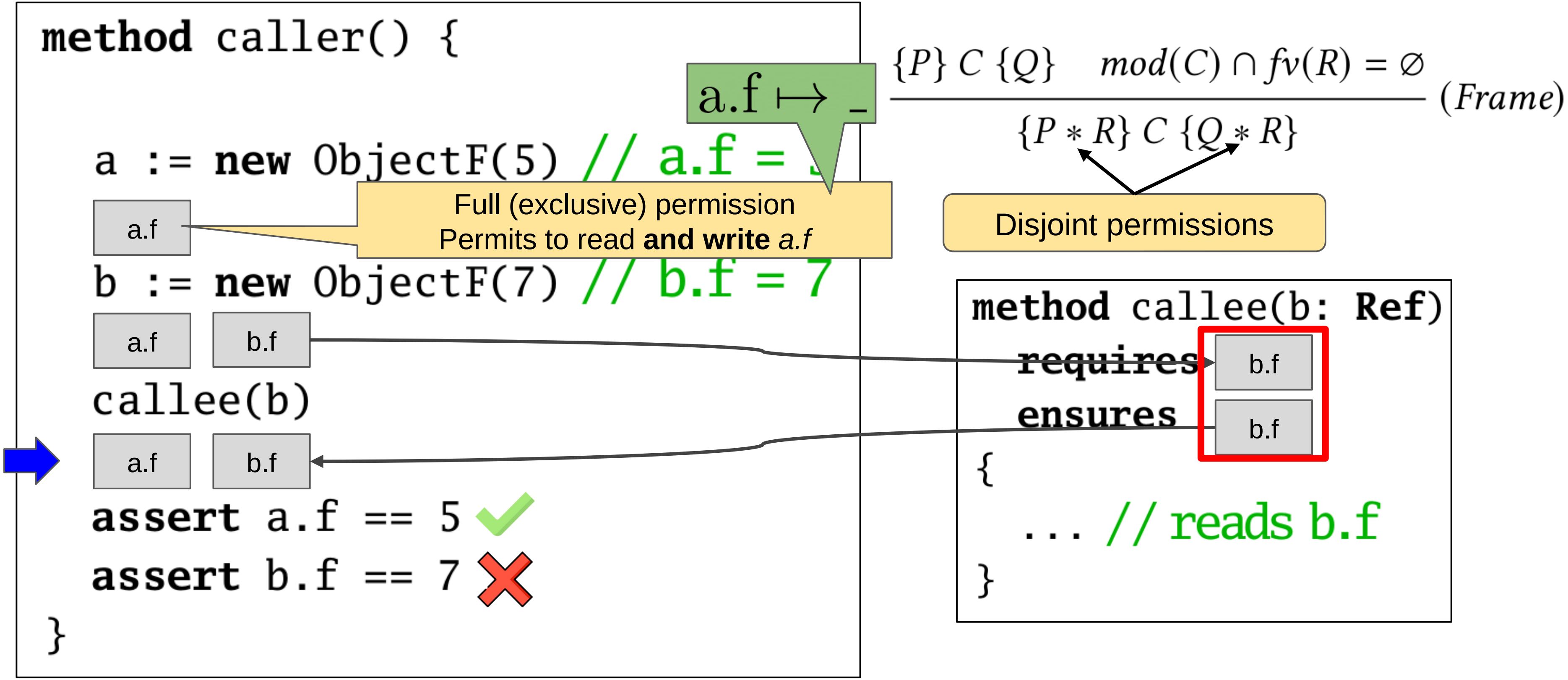
{
  ... // reads b.f
}
```

Separation logic



Fractional permissions

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  a.f
  b := new ObjectF(7) // b.f = 7
  a.f   b.f
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

```
method callee(b: Ref)
  requires
  ensures
{
  ... // reads b.f
}
```

Fractional permissions

```
method caller() {

   a := new ObjectF(5) // a.f = 5
   [a.f]
   b := new ObjectF(7) // b.f = 7
   [a.f] [b.f]
   callee(b)

   assert a.f == 5 ✔
   assert b.f == 7 ✘
}
```

```
method callee(b: Ref)
   requires [b.f]
   ensures [b.f]
{
   ... // reads .f
}
```

$b.f \overset{0.5}{\mapsto} \_$

Fractional (non-exclusive) permission
Permits only to read *b.f*

Fractional permissions

```
method caller() {

  a := new ObjectF(5) // a.f = 5
```

a.f

```
  b := new ObjectF(7) // b.f = 7
```

a.f b.f b.f

```
  callee(b)

  assert a.f == 5
  assert b.f == 7
}
```

```
method callee(b: Ref)
  requires
  ensures
{
  ... // reads .f
}
```

requires: b.f

ensures: b.f — $b.f \overset{0.5}{\mapsto} \_$

Fractional (non-exclusive) permission
Permits only to read *b.f*

Fractional permissions

```
method caller() {

  a := new ObjectF(5) // a.f = 5
  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f | b.f]
  callee(b)
  [a.f] [b.f]
  assert a.f == 5  ✔
  assert b.f == 7  ✘
}
```

```
method callee(b: Ref)
  requires [b.f]
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  callee(b)
  [a.f] [b.f | b.f]
  assert a.f == 5 ✔
  assert b.f == 7 ✔
}
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method callee(b: Ref)
  requires [b.f]
  ensures [b.f]
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  ... // reads b.f
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Fractional (non-exclusive) permission
Permits only to read *b.f*

Fractional permissions

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  [a.f]
  b := new ObjectF(7) // b.f = 7
  [a.f] [b.f | b.f]
  callee(b)
  [a.f] [b.f | b.f]
  assert a.f == 5 ✓
  assert b.f == 7 ✓
}
```

$$\text{State} \triangleq \text{Locations} \rightharpoonup \text{Values} \times (\mathbb{Q} \cap (0,1])$$

```
method callee(b: Ref)
  requires b.f
  ensures b.f
{
  ... // reads .f
}
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$b.f \overset{0.5}{\mapsto} \_$

Fractional (non-exclusive) permission
Permits only to read *b.f*

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   assert b.f == 7 ✔
}
```

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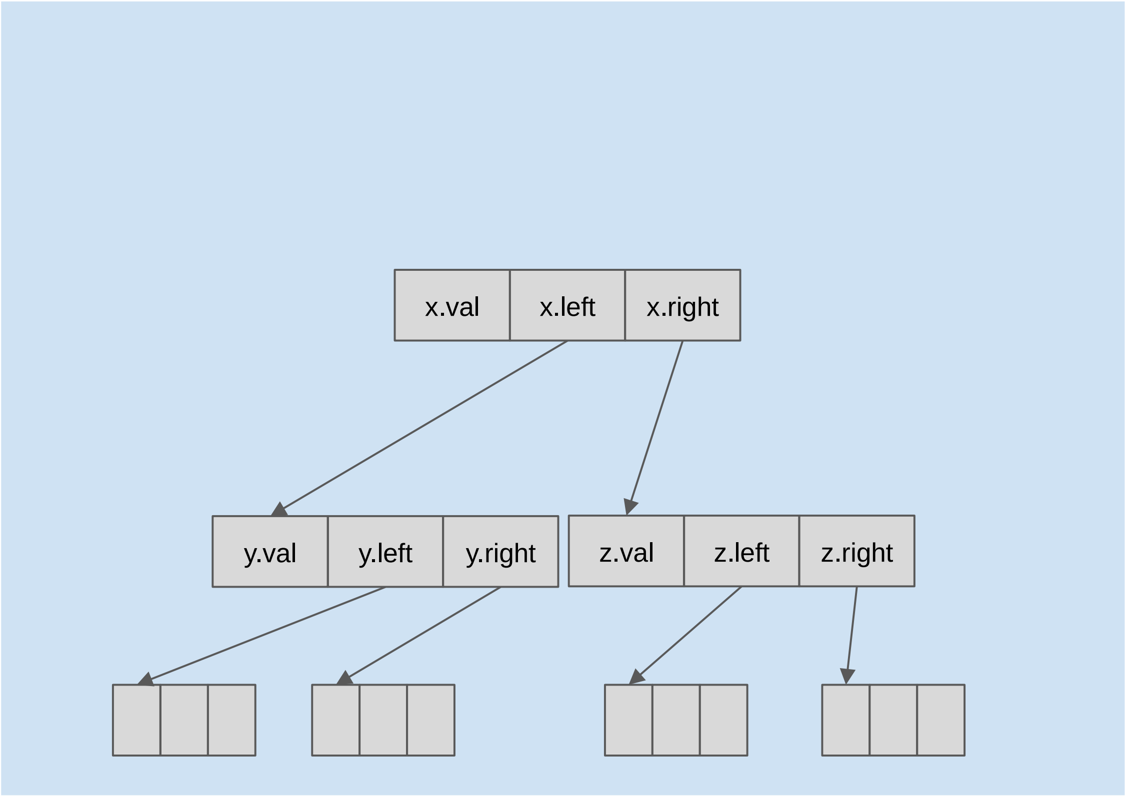
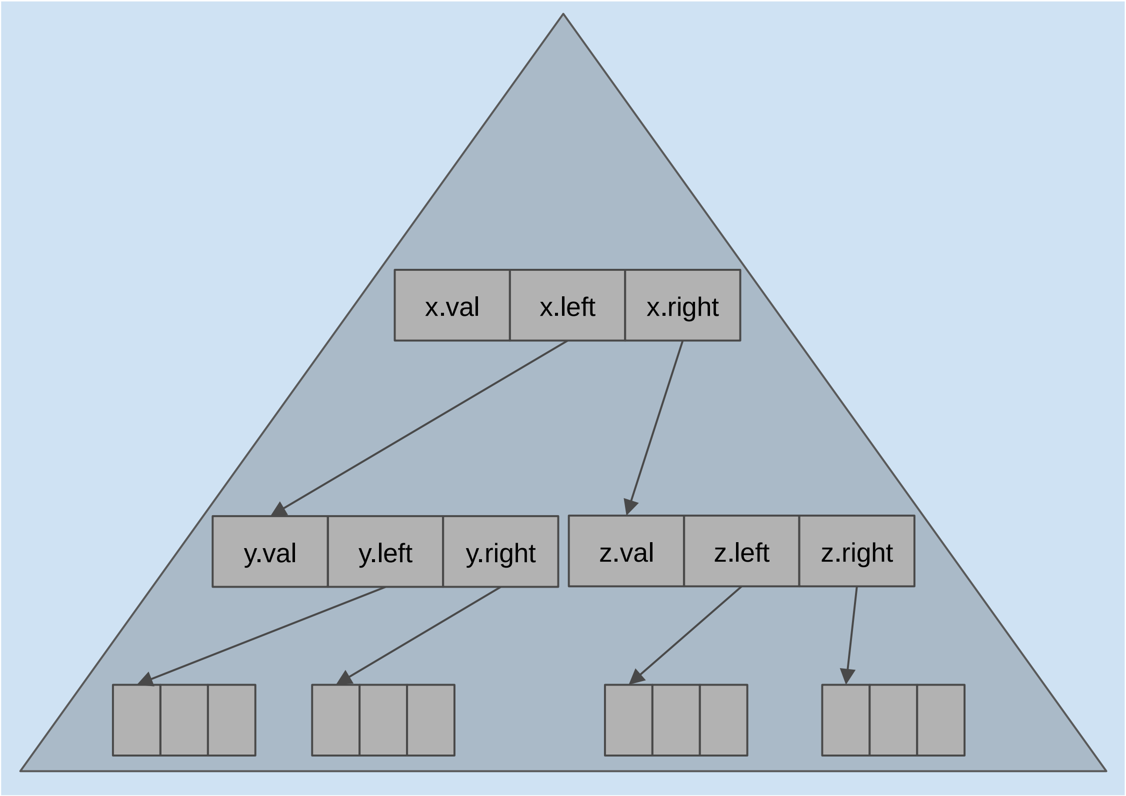
Fractional (non-exclusive) permission
Permits only to read *b.f*

$$\text{State} \triangleq \text{Locations} \rightharpoonup \text{Values} \times (\mathbb{Q} \cap (0,1])$$

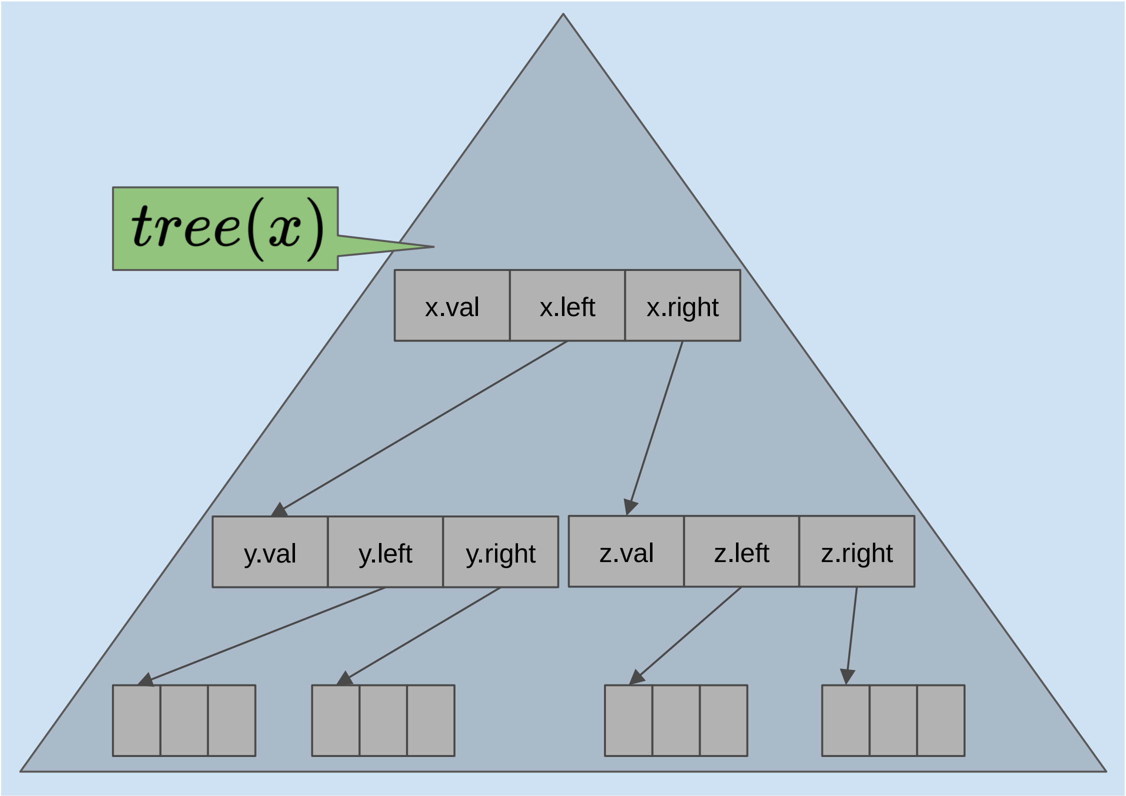
(Fractional) resources, informally



(Fractional) resources, informally

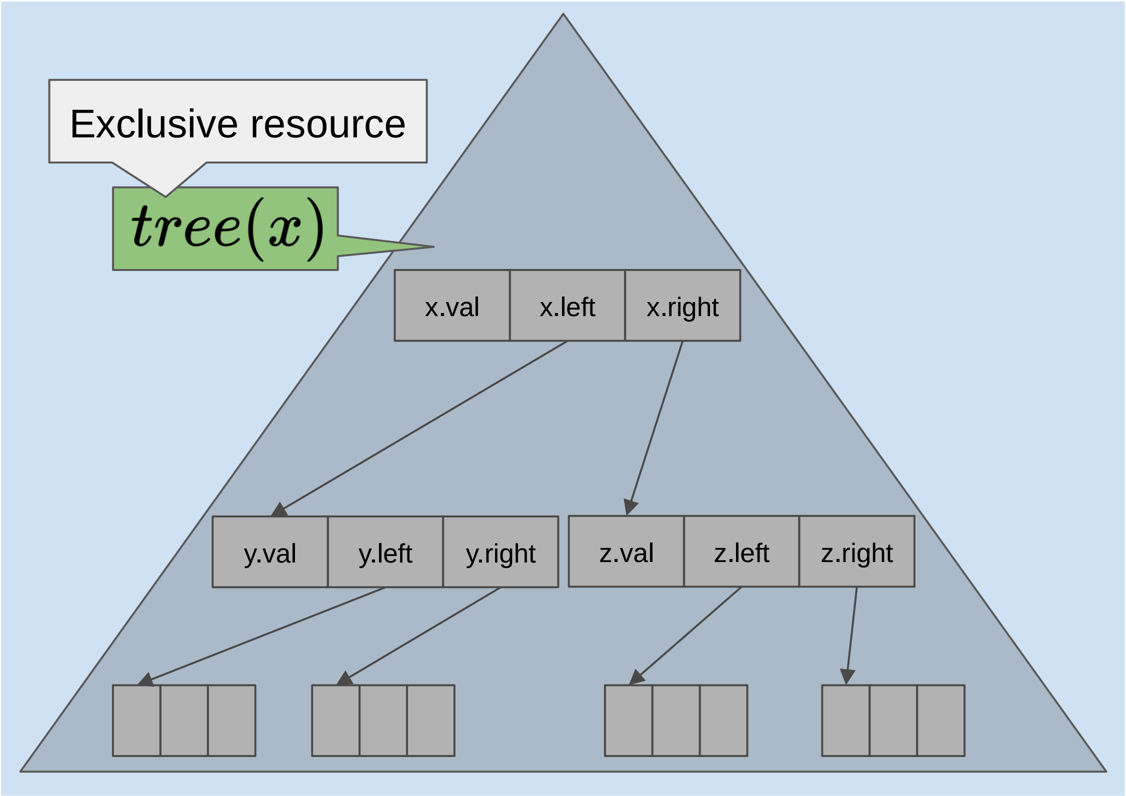


(Fractional) resources, informally



$$tree(x) \triangleq (x \neq \texttt{null} \Rightarrow x.val \mapsto \_ *$$
$$(\exists x_l.\, x.left \mapsto x_l * tree(x_l)) * (\exists x_r.\, x.right \mapsto x_r * tree(x_r)))$$

(Fractional) resources, informally



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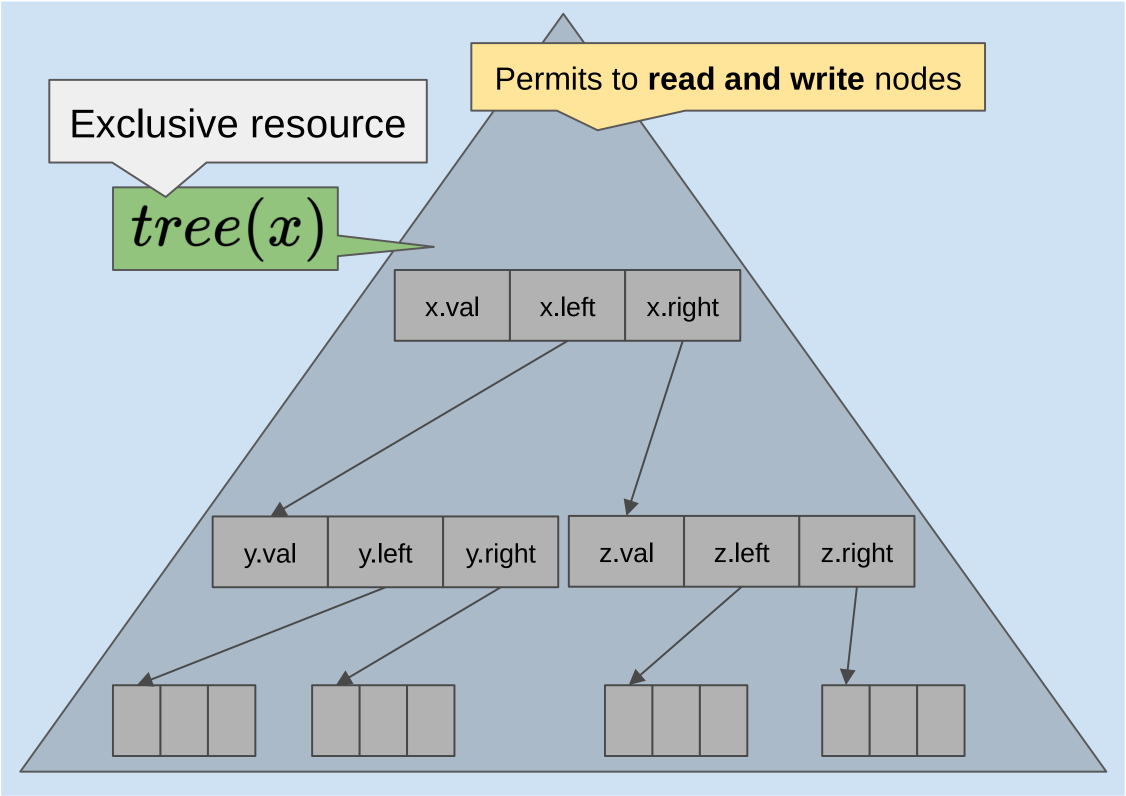

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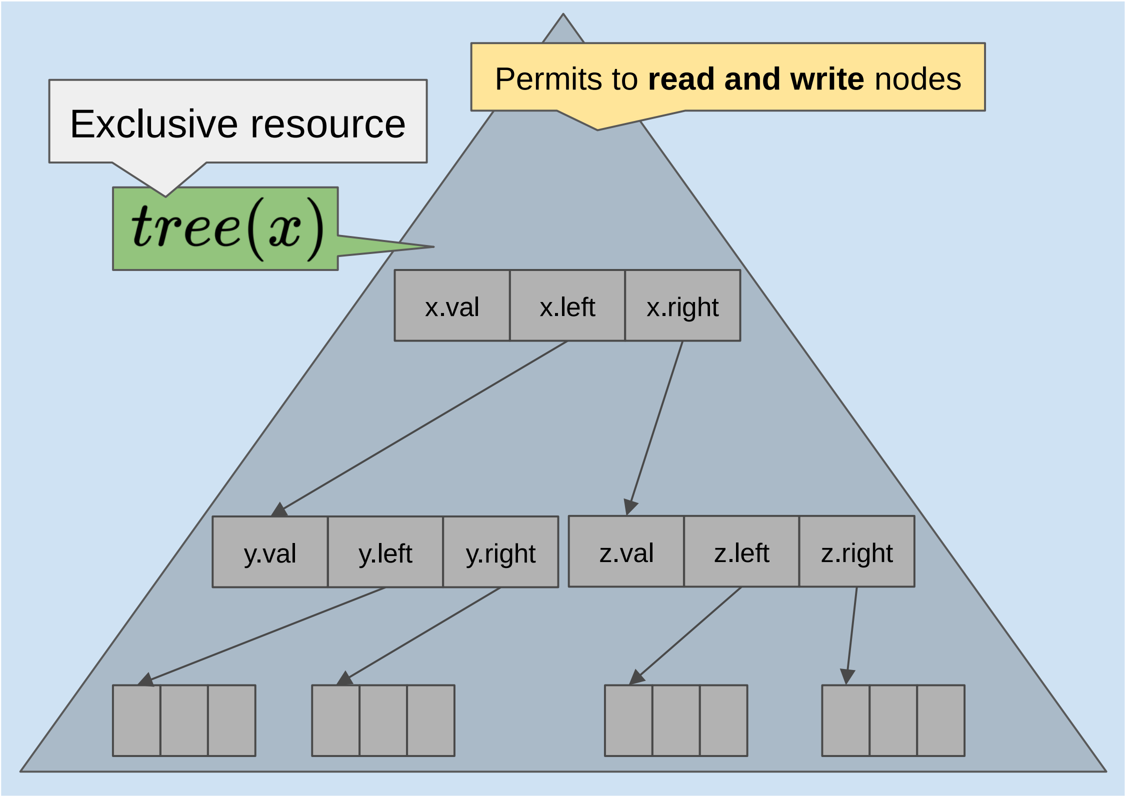

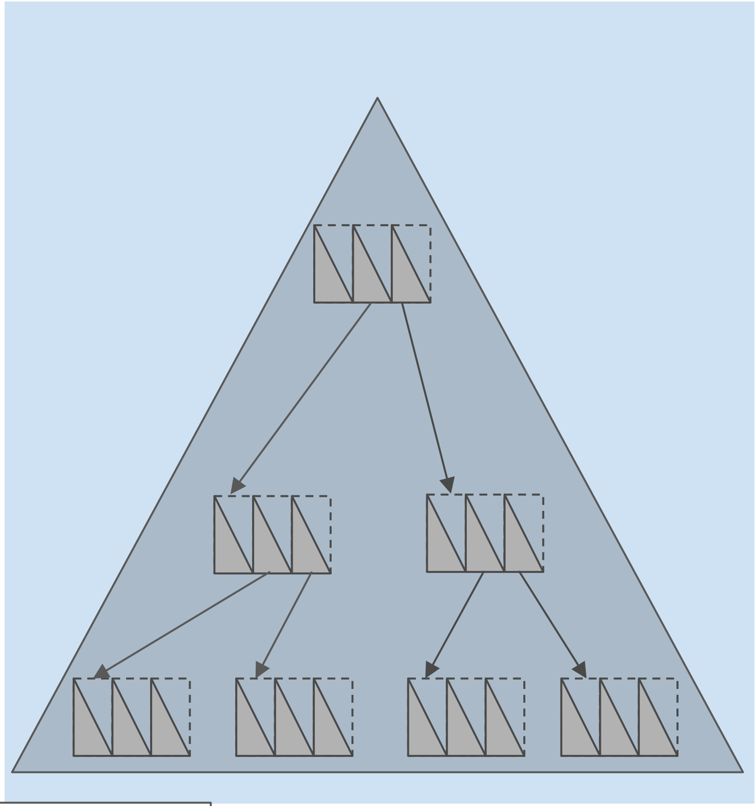

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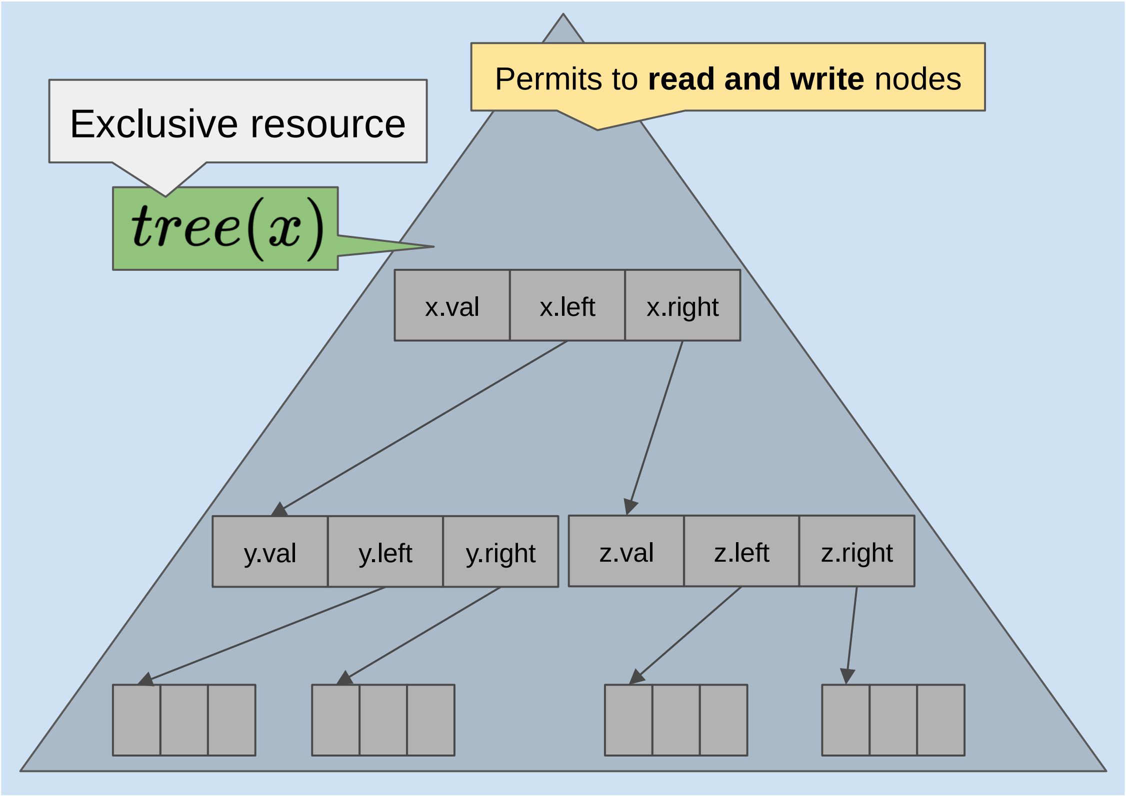
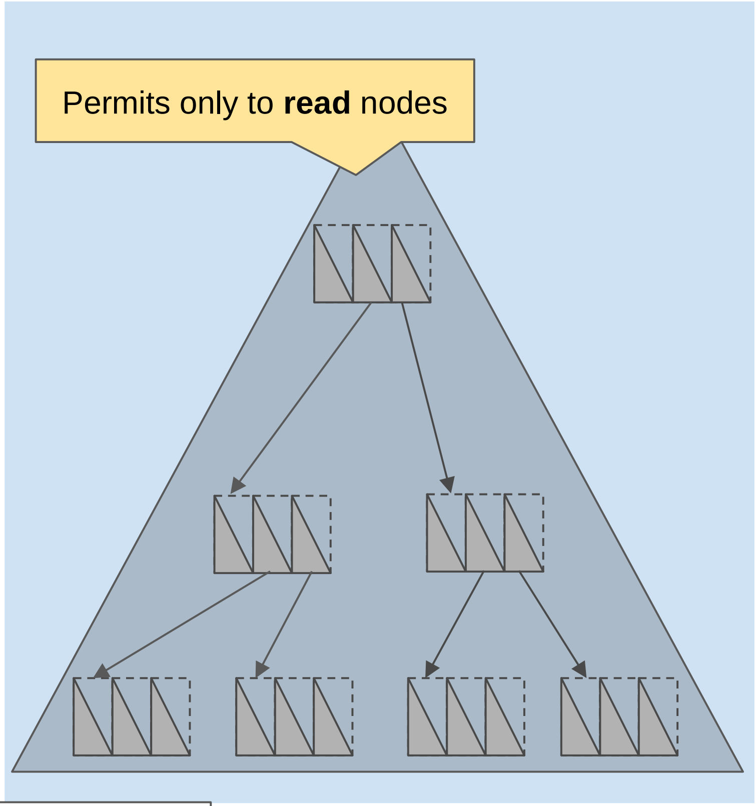
(Fractional) resources, informally





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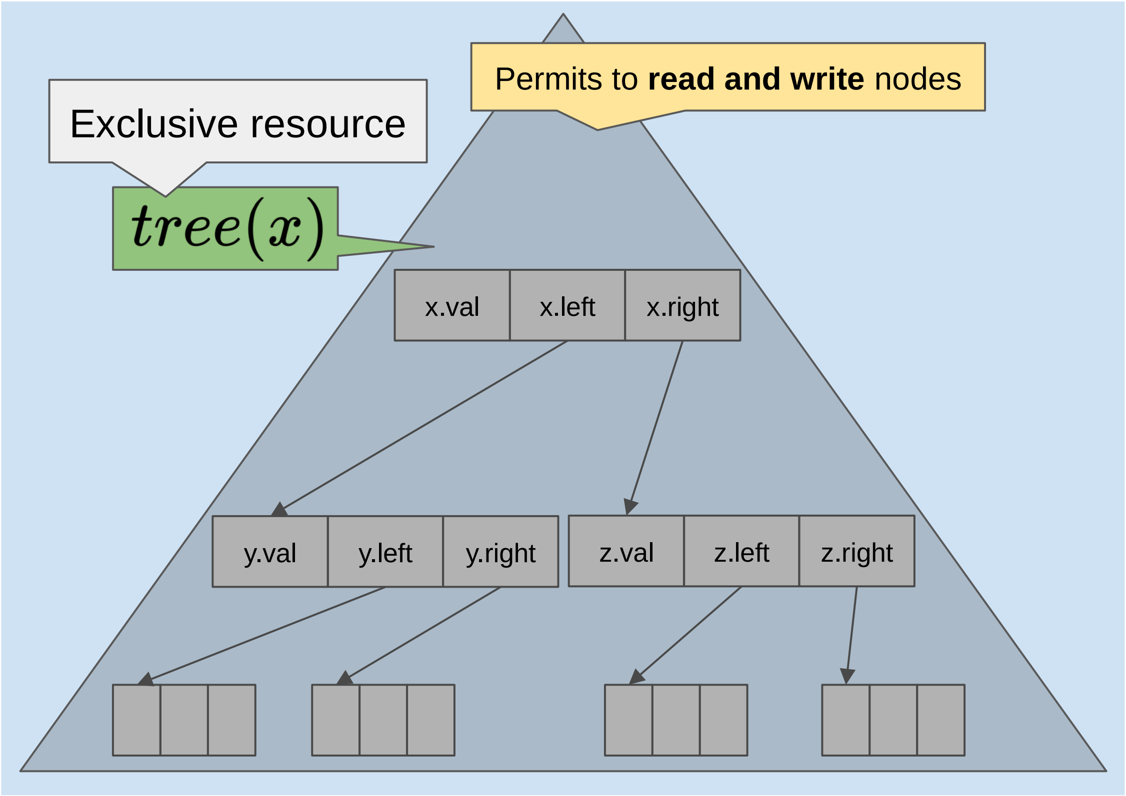
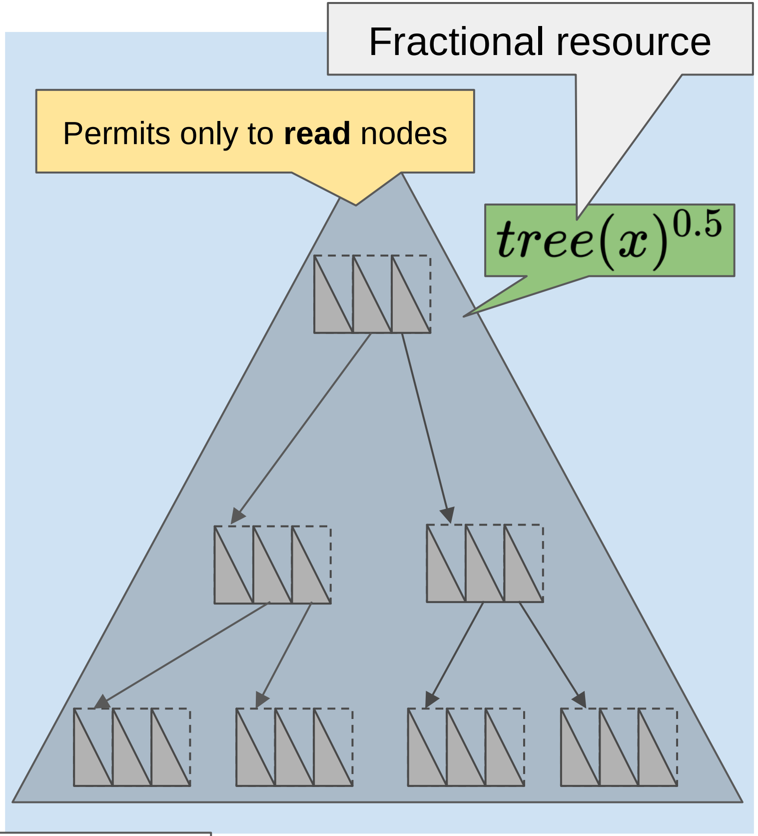
$$(\exists x_l.\, x.left \mapsto x_l * tree(x_l)) * (\exists x_r.\, x.right \mapsto x_r * tree(x_r)))$$

(Fractional) resources, informally





$$tree(x) \triangleq (x \neq \texttt{null} \Rightarrow x.val \mapsto \_ *$$
$$(\exists x_l.\, x.left \mapsto x_l * tree(x_l)) * (\exists x_r.\, x.right \mapsto x_r * tree(x_r)))$$

Using fractional resources

```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {
          print(x.val)                    ||    print(x.val)
          processTree(x.left)             ||    processTree(x.left)
          processTree(x.right)            ||    processTree(x.right)
  }
  {tree(x)^π}
}
```

Using fractional resources

```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {

          print(x.val)                 ||    print(x.val)
          processTree(x.left)          ||    processTree(x.left)
          processTree(x.right)         ||    processTree(x.right)

  }
  {tree(x)^π}
}
```

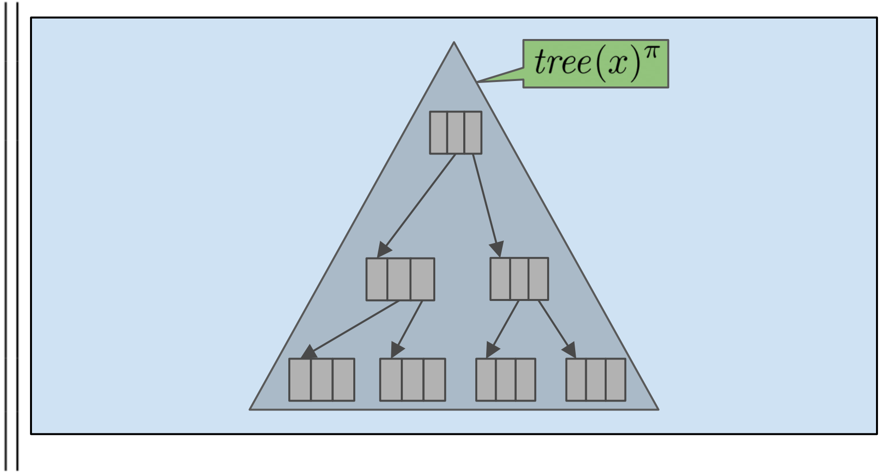
Taken from "Logical Reasoning for Disjoint Permissions", Xuan-Bach Le and Aquinas Hobor (ESOP'18)

Using fractional resources

```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {

              print(x.val)
              processTree(x.left)
              processTree(x.right)

  }
  {tree(x)^π}
}
```



Using fractional resources

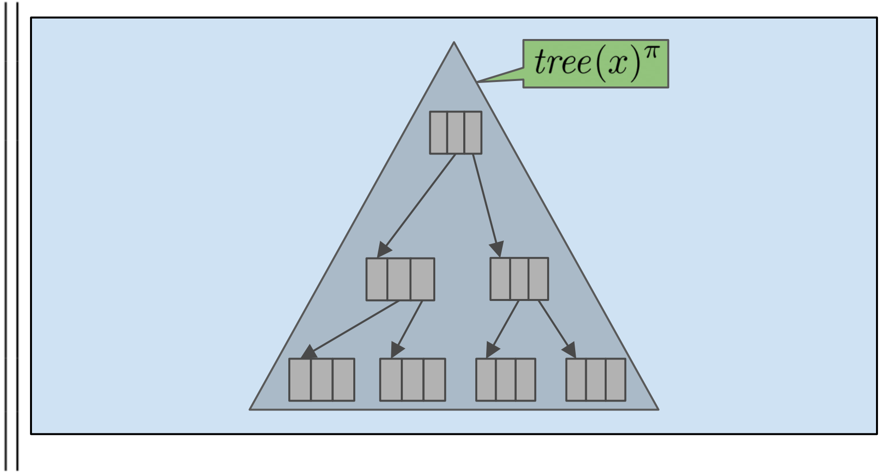
```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {
    {tree(x)^π * x ≠ null}



              print(x.val)
              processTree(x.left)
              processTree(x.right)



  }
  {tree(x)^π}
}
```



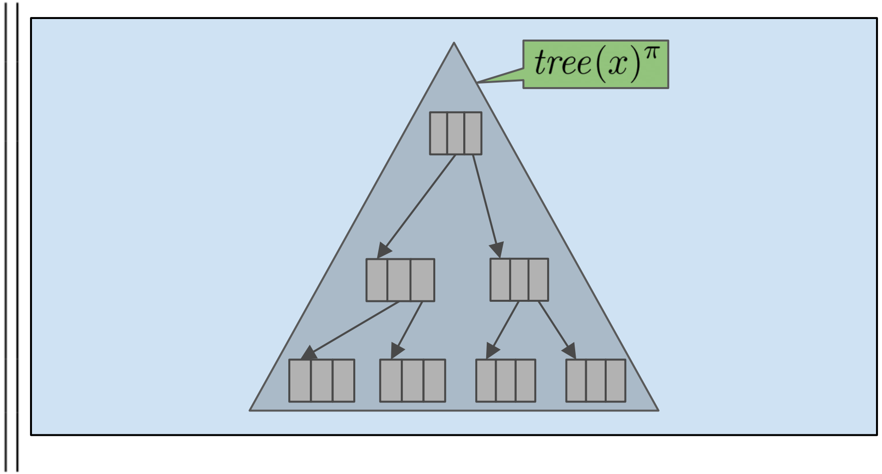
Using fractional resources

```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {
    {tree(x)^π * x ≠ null}

        print(x.val)
        processTree(x.left)
        processTree(x.right)

  }
  {tree(x)^π}
}
```

$$\frac{\{P_1\}\ C_1\ \{Q_1\} \quad \{P_2\}\ C_2\ \{Q_2\}}{\{P_1 * P_2\}\ C_1 \parallel C_2\ \{Q_1 * Q_2\}}\ (Parallel)$$



Using fractional resources

```
method processTree(x: Ref) {
```

$\{tree(x)^{\pi}\}$

```
  if (x != null) {
```

$\{tree(x)^{\pi} * x \neq null\}$

$\{(tree(x)^{\frac{\pi}{2}} * x \neq null) * (tree(x)^{\frac{\pi}{2}} * x \neq null)\}$

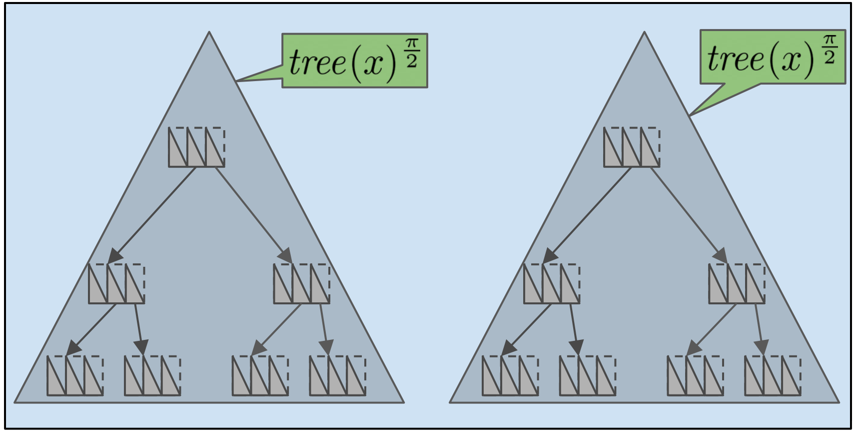
1. Split

```
        print(x.val)
        processTree(x.left)
        processTree(x.right)
  }
```

$\{tree(x)^{\pi}\}$

```
}
```

$$\frac{\{P_1\}\ C_1\ \{Q_1\} \quad \{P_2\}\ C_2\ \{Q_2\}}{\{P_1 * P_2\}\ C_1 \parallel C_2\ \{Q_1 * Q_2\}}\ (Parallel)$$



Using fractional resources

method processTree(x: **Ref**) {

$\{tree(x)^{\pi}\}$

if (x != **null**) {

$\{tree(x)^{\pi} * x \neq null\}$

$\{(tree(x)^{\frac{\pi}{2}} * x \neq null) * (tree(x)^{\frac{\pi}{2}} * x \neq null)\}$

1. Split

$\{tree(x)^{\frac{\pi}{2}} * x \neq null\}$

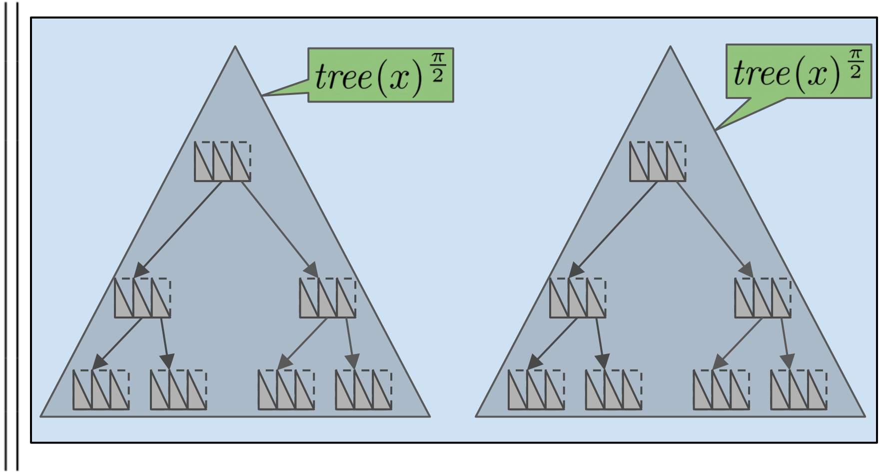
```
print(x.val)
processTree(x.left)
processTree(x.right)
```

}

$\{tree(x)^{\pi}\}$

}

$$\frac{\{P_1\}\ C_1\ \{Q_1\} \qquad \{P_2\}\ C_2\ \{Q_2\}}{\{P_1 * P_2\}\ C_1 \parallel C_2\ \{Q_1 * Q_2\}}\ (Parallel)$$



Using fractional resources

```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {
    {tree(x)^π * x ≠ null}
    {(tree(x)^(π/2) * x ≠ null) * (tree(x)^(π/2) * x ≠ null)}
```

1. Split

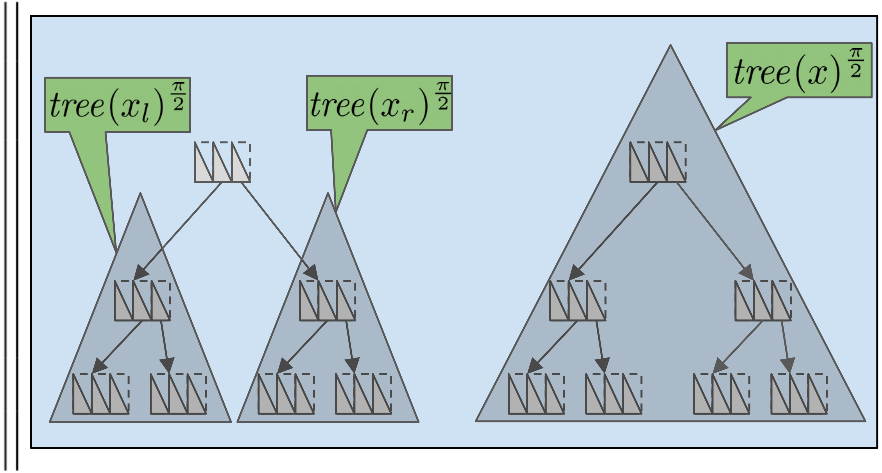
$\{tree(x)^{\frac{\pi}{2}} * x \neq null\}$

$\{x.val \overset{\frac{\pi}{2}}{\mapsto} \_ * \ldots * tree(x_l)^{\frac{\pi}{2}} * tree(x_r)^{\frac{\pi}{2}}\}$

2. Distribute

```
    print(x.val)
    processTree(x.left)
    processTree(x.right)
  }
  {tree(x)^π}
}
```

$$\frac{\{P_1\}\ C_1\ \{Q_1\} \qquad \{P_2\}\ C_2\ \{Q_2\}}{\{P_1 * P_2\}\ C_1 \parallel C_2\ \{Q_1 * Q_2\}}\ (Parallel)$$



Using fractional resources

```
method processTree(x: Ref) {
```

$\{tree(x)^{\pi}\}$

```
  if (x != null) {
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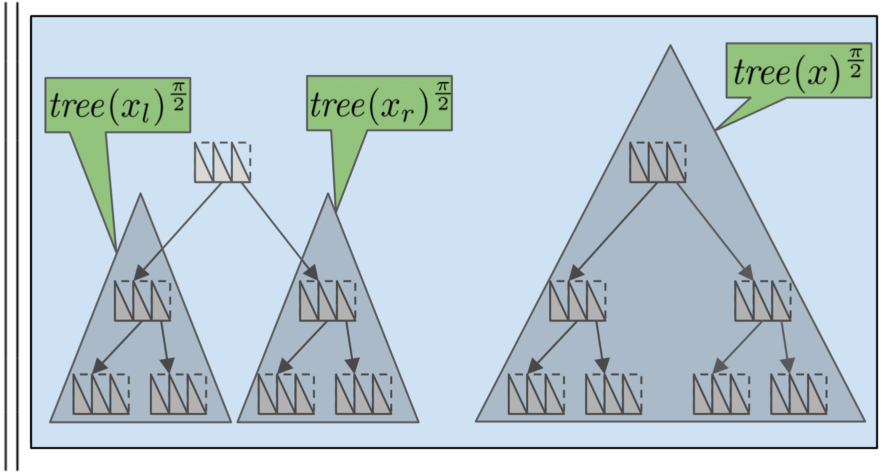

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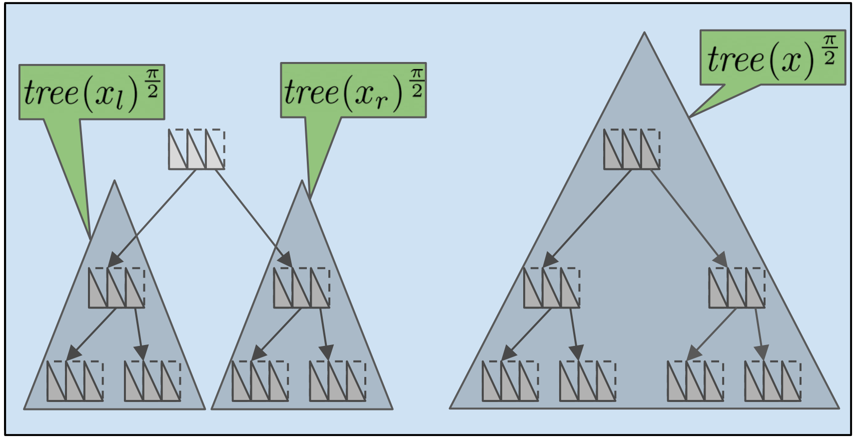
||



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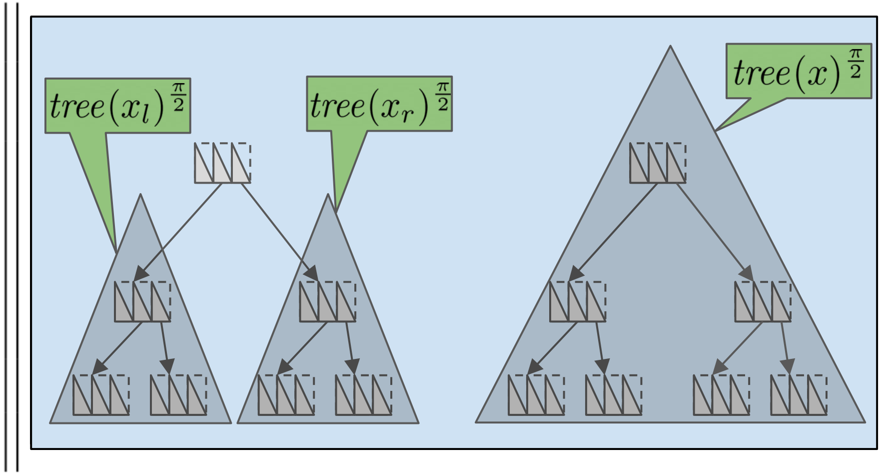

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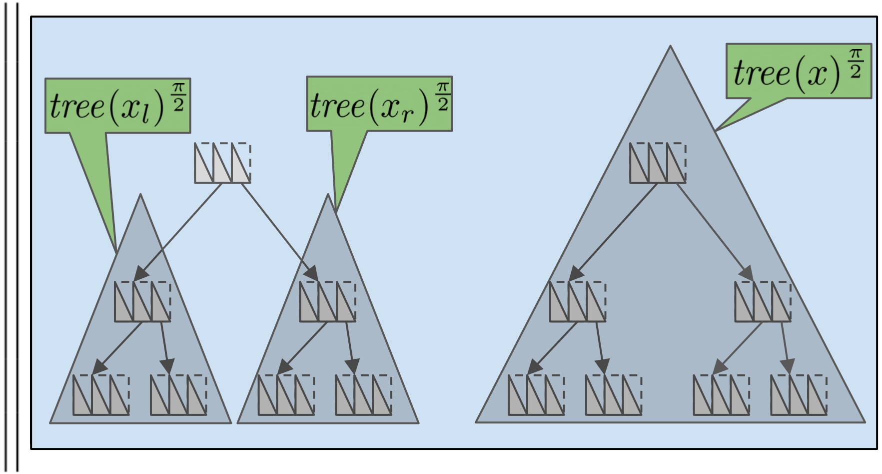

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3. Factorise

```
  }
```

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```
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```

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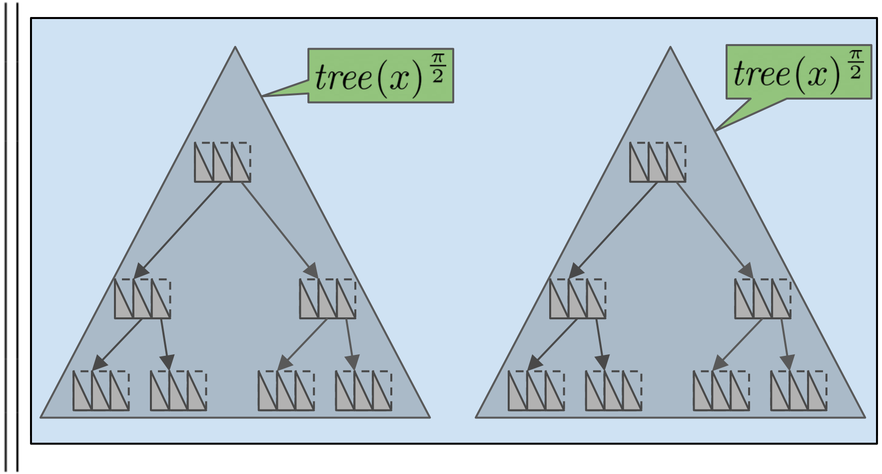

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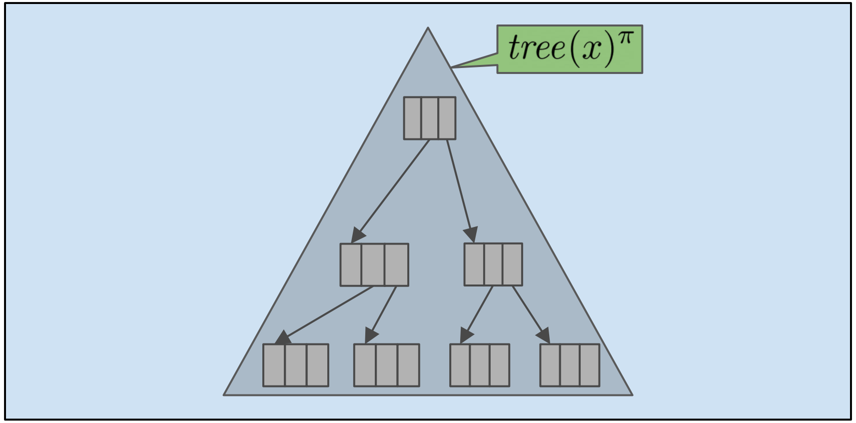

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```
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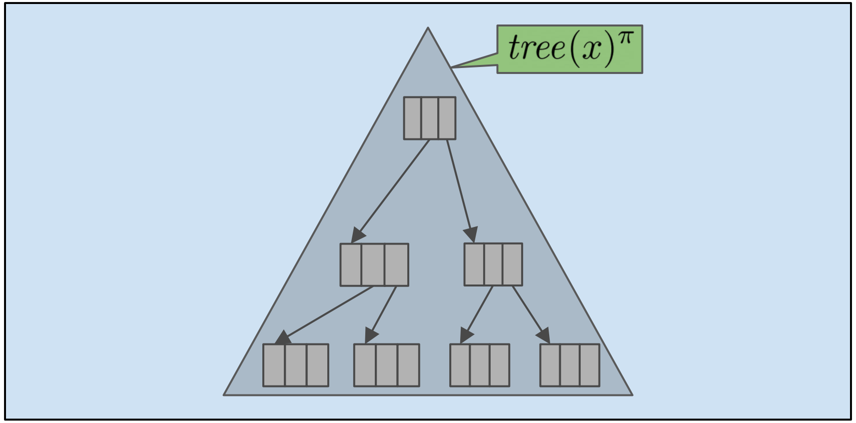

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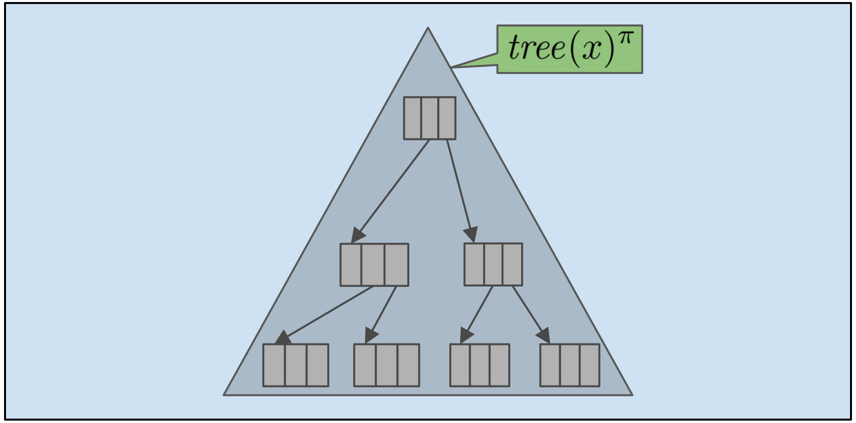

Is this proof outline actually correct?

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It depends on the meaning of fractional resources.

The meaning(s) of fractional resources

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Semantic multiplication

Studied in theoretical papers

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Previous proof outline ❌

The meaning(s) of fractional resources

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$$A^{\pi}$$

Previous proof outline ✗

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$$\text{State} \triangleq \text{Locations} \rightharpoonup \text{Values} \times (\mathbb{Q} \cap (0,1])$$

$$h \models A^{\pi} \text{ iff there exists } h_A \text{ such that}$$
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Previous proof outline ✗

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All permission amounts are multiplied by π

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Implemented in automatic separation logic verifiers (e.g, VeriFast, Viper…)

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$$0.5 \cdot (l_1 \mapsto v_1 * l_2 \mapsto v_2)$$
$$\triangleq 0.5 \cdot (l_1 \mapsto v_1) * 0.5 \cdot (l_2 \mapsto v_2)$$
$$\triangleq (l_1 \overset{0.5}{\mapsto} v_1) * (l_2 \overset{0.5}{\mapsto} v_2)$$

Previous proof outline ✓

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  - **Syntactic** multiplication: Rules implemented in automated verifiers, no formal foundation
  - **Semantic** multiplication: Theoretical foundation, shortcomings
- We present and formalise a new logic: **unbounded separation logic**
  - Formal foundation for the **syntactic** multiplication
  - Eliminates shortcomings from the **semantic** multiplication
- In-depth study of **combinability** in unbounded separation logic
- Reasoning principles for (co)inductive predicates
- *Unbounded separation logic* as a formal foundation for automatic verifiers
  - Justifies the rules used
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Semantic multiplication ≠ syntactic multiplication (1/2)

$$tree(x)^{0.5}$$

Semantic

$$0.5 \cdot tree(x)$$

Syntactic

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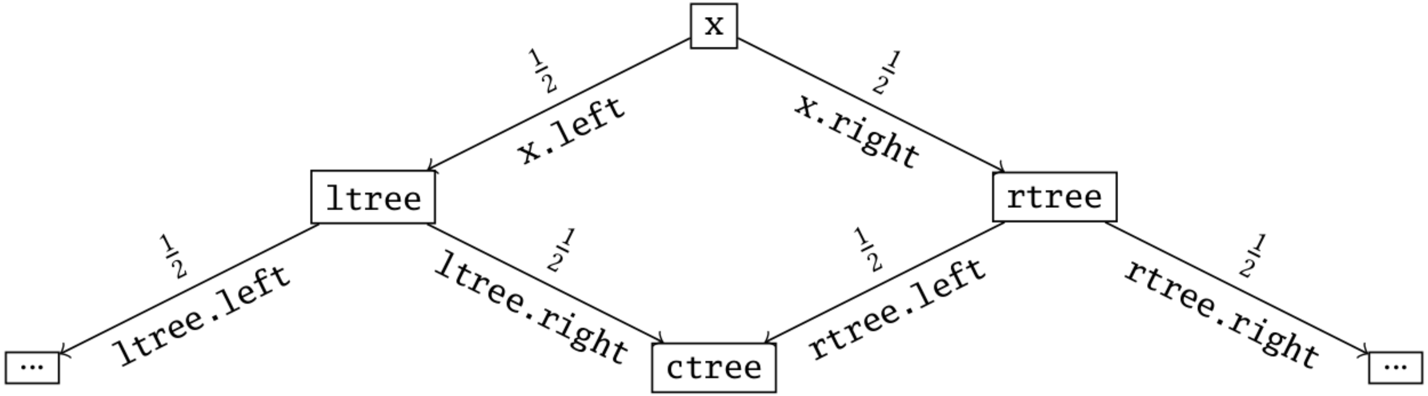

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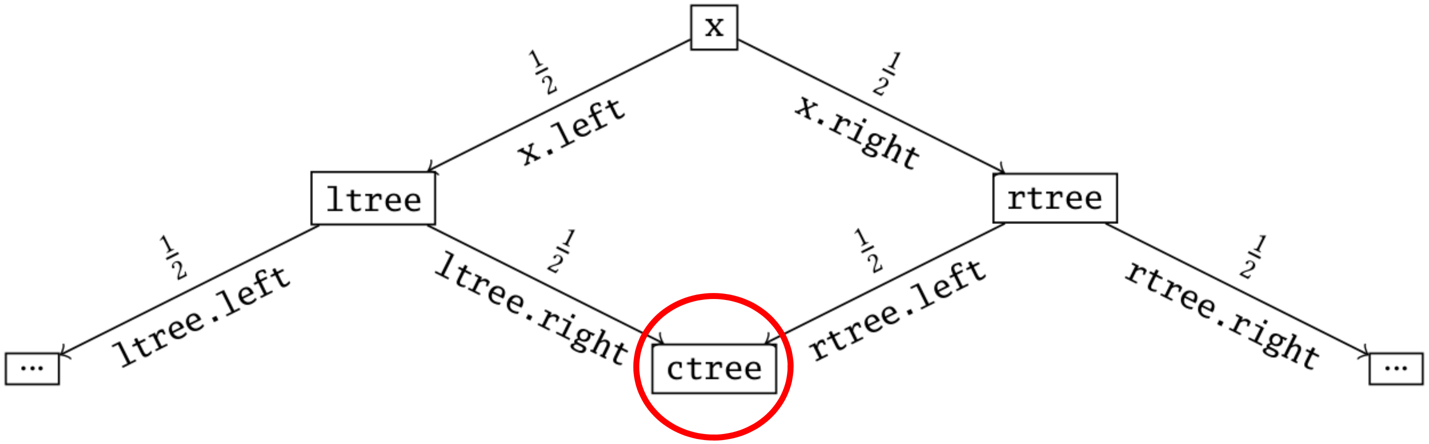

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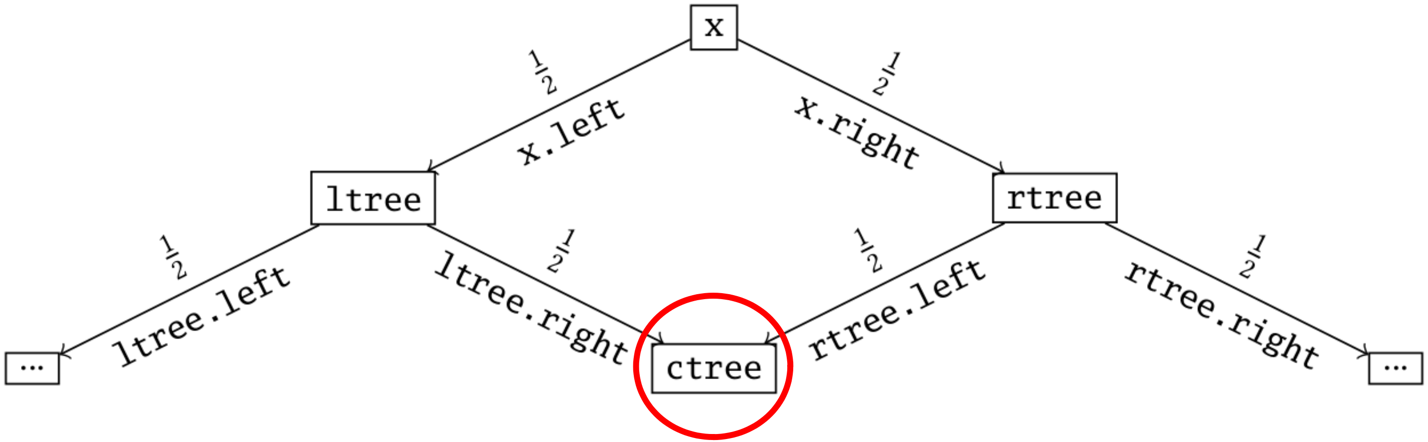

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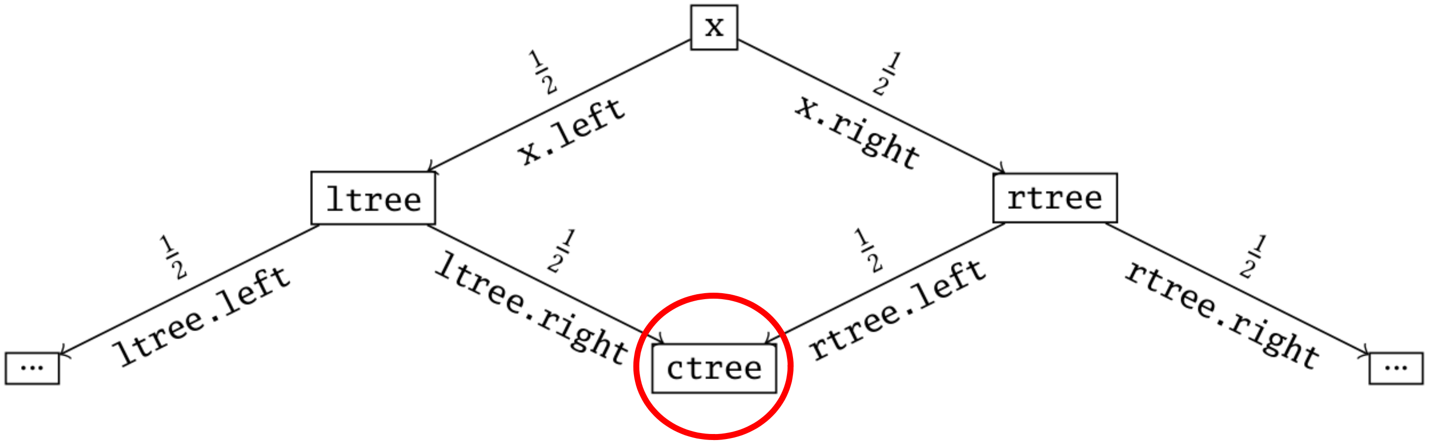

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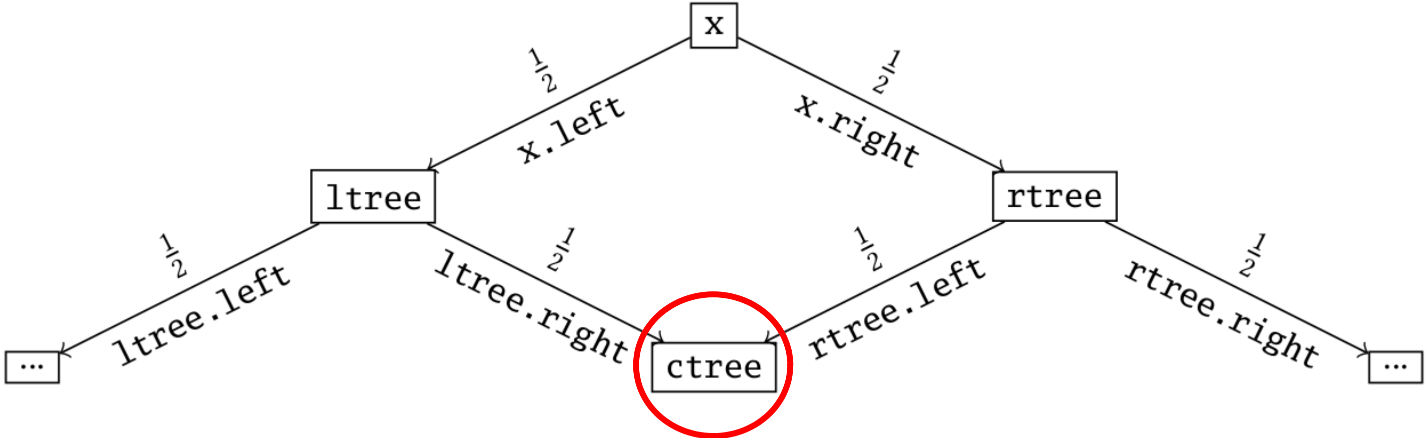

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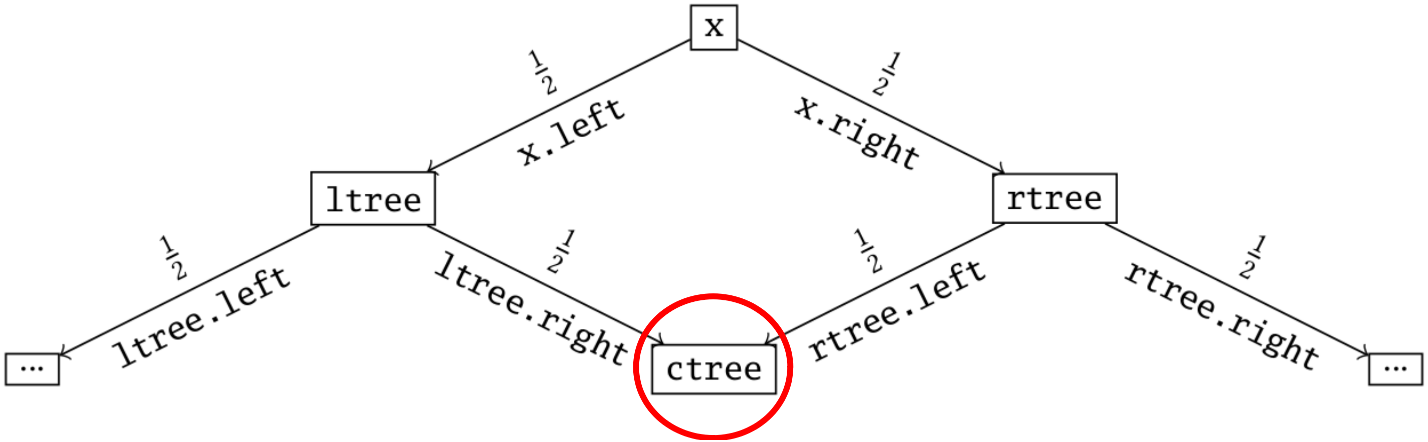

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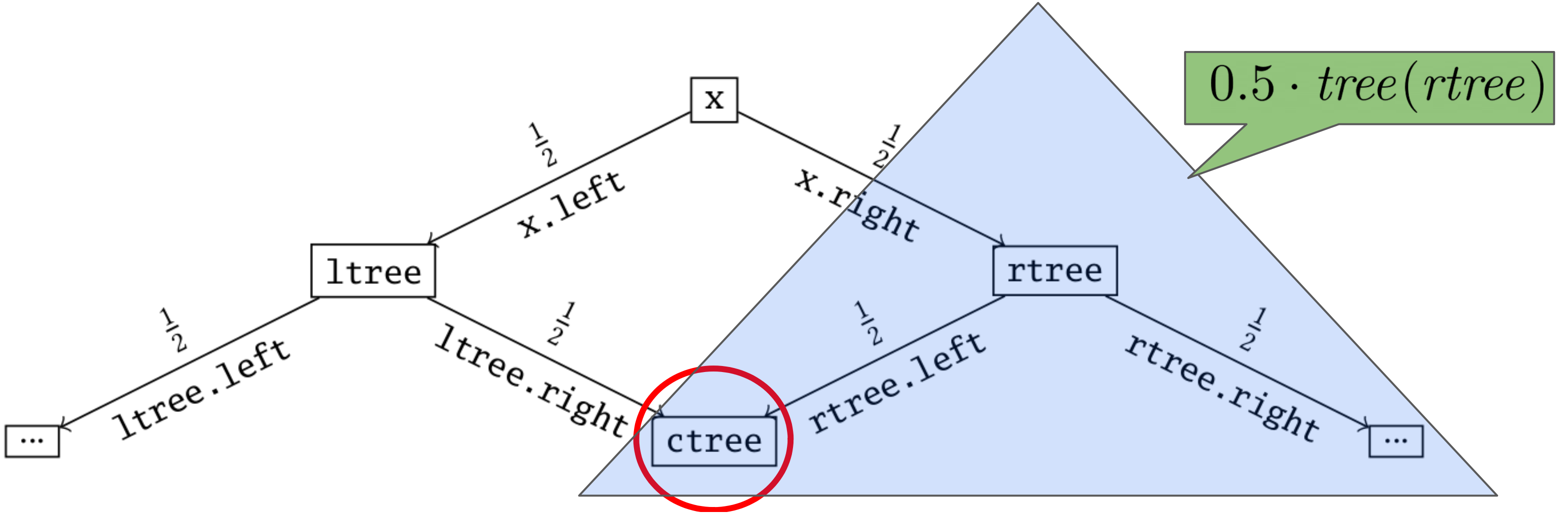

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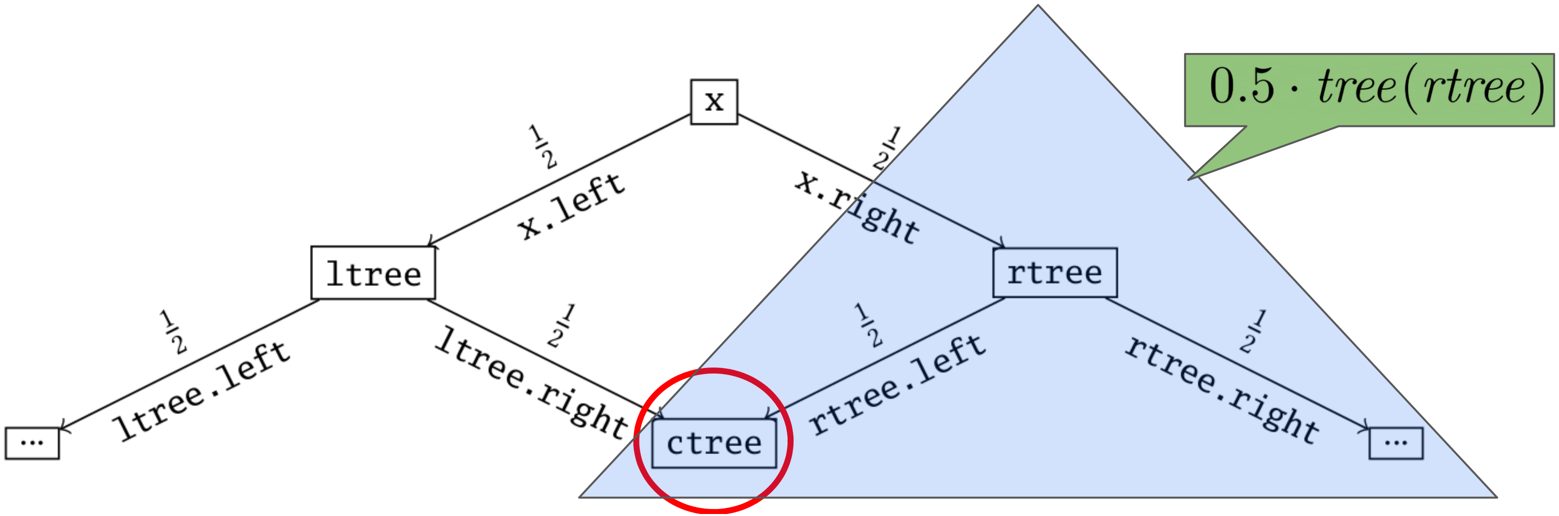

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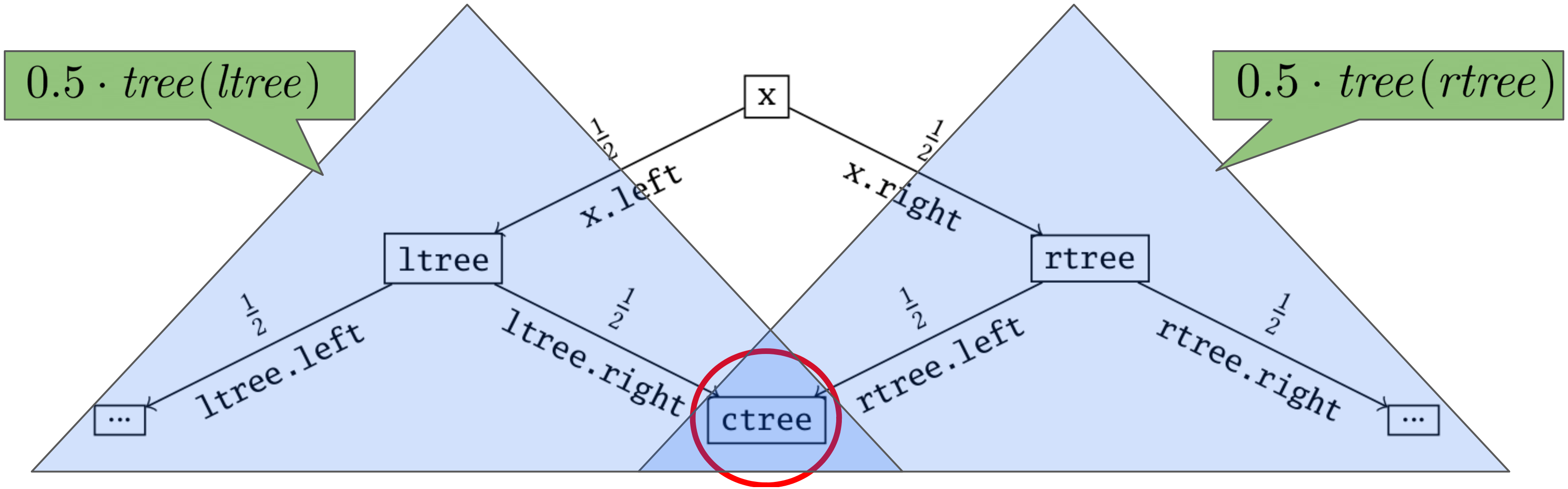

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Syntactic — Syntactic — Syntactic

Semantic multiplication ≠ syntactic multiplication (2/2)

Syntactic

Semantic

$$0.5 \cdot (l_1 \mapsto v_1 * l_2 \mapsto v_2) \not\models (l_1 \mapsto v_1 * l_2 \mapsto v_2)^{0.5}$$

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Semantic multiplication ≠ syntactic multiplication (2/2)

Syntactic

l_1 and l_2 **cannot** be aliases

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l_1 and l_2 **can** be aliases

$$(l_1 \mapsto v_1)^{0.5} * (l_2 \mapsto v_2)^{0.5}$$

Semantic

Semantic

Cannot factorise!

$$A^\pi * B^\pi \not\models (A * B)^\pi$$

Summary

	Semantic multiplication	Syntactic multiplication
Factorisability ($*$)		
Distributivity ($*$)		

Factorise

$$A^\pi * B^\pi \longrightarrow (A * B)^\pi$$

Summary

	Semantic multiplication	**Syntactic multiplication**
Factorisability ($*$)	❌	
Distributivity ($*$)		

$$A^\pi * B^\pi \xrightarrow{\text{Factorise}} (A * B)^\pi$$

Summary

	Semantic multiplication	**Syntactic multiplication**
Factorisability ($*$)	✗	
Distributivity ($*$)		

Factorise →

$$A^{\pi} * B^{\pi} \rightleftarrows (A * B)^{\pi}$$

← Distribute

Summary

	Semantic multiplication	**Syntactic multiplication**
Factorisability ($*$)	✗	
Distributivity ($*$)	✓	

$$A^{\pi} * B^{\pi} \underset{\text{Distribute}}{\overset{\text{Factorise}}{\rightleftarrows}} (A * B)^{\pi}$$

Summary

	Semantic multiplication	Syntactic multiplication
Factorisability (*)	✗	✓
Distributivity (*)	✓	✓

$$A^{\pi} * B^{\pi} \underset{\text{Distribute}}{\overset{\text{Factorise}}{\rightleftarrows}} (A * B)^{\pi}$$

Summary

	Semantic multiplication	Syntactic multiplication
Factorisability (∗)	✗	✓
Distributivity (∗)	✓	✓
Factorisability (– ∗)		
Distributivity (– ∗)		

Separating implication (magic wand)

$$A^{\pi} * B^{\pi} \underset{\text{Distribute}}{\overset{\text{Factorise}}{\rightleftarrows}} (A * B)^{\pi}$$

Summary

	Semantic multiplication	**Syntactic multiplication**
Factorisability ($*$)	✗	✓
Distributivity ($*$)	✓	✓
Factorisability ($-*$)	✓	
Distributivity ($-*$)	✗	

Separating implication (magic wand)

Factorise

$$A^\pi * B^\pi \rightleftarrows (A * B)^\pi$$

Distribute

Summary

has shortcomings

	Semantic multiplication	**Syntactic multiplication**
Factorisability (∗)	✗	✓
Distributivity (∗)	✓	✓
Factorisability (–∗)	✓	
Distributivity (–∗)	✗	

Separating implication (magic wand)

Factorise

$$A^\pi * B^\pi \rightleftarrows (A * B)^\pi$$

Distribute

Summary

has shortcomings

	Semantic multiplication	**Syntactic multiplication**
Factorisability (∗)	✗	✓
Distributivity (∗)	✓	✓
Factorisability (–∗)	✓	?
Distributivity (–∗)	✗	?

Unsupported

Separating implication (magic wand)

Factorise

$$A^{\pi} * B^{\pi} \rightleftarrows (A * B)^{\pi}$$

Distribute

Summary

has shortcomings

no theoretical foundation

	Semantic multiplication	**Syntactic multiplication**
Factorisability ($*$)	✗	✓
Distributivity ($*$)	✓	✓
Factorisability ($-*$)	✓	?
Distributivity ($-*$)	✗	?

Unsupported

Separating implication (magic wand)

Factorise

$$A^{\pi} * B^{\pi} \rightleftarrows (A * B)^{\pi}$$

Distribute

Unbounded separation logic

has shortcomings

no theoretical foundation

	Semantic multiplication	Syntactic multiplication
Factorisability (*)	✗	✓
Distributivity (*)	✓	✓
Factorisability (–*)	✓	?
Distributivity (–*)	✗	?

In **bounded** separation logic

Unbounded separation logic

has shortcomings

no theoretical foundation

	Semantic multiplication	Syntactic multiplication
Factorisability (∗)	✗	✓
Distributivity (∗)	✓	✓
Factorisability (−∗)	✓	?
Distributivity (−∗)	✗	?

In **bounded** separation logic

(Syntactic) multiplication
✓
✓
✓
✓

In **unbounded** separation logic

provides a theoretical foundation

Unbounded separation logic: Intuition

Semantic

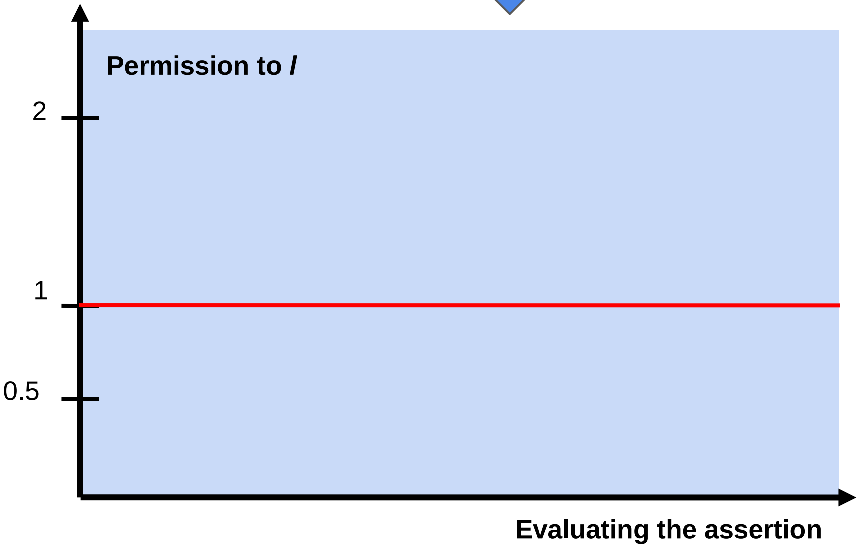
$$0.5 \cdot (l \mapsto v * l \mapsto v) \not\models (l \mapsto v * l \mapsto v)^{0.5}$$

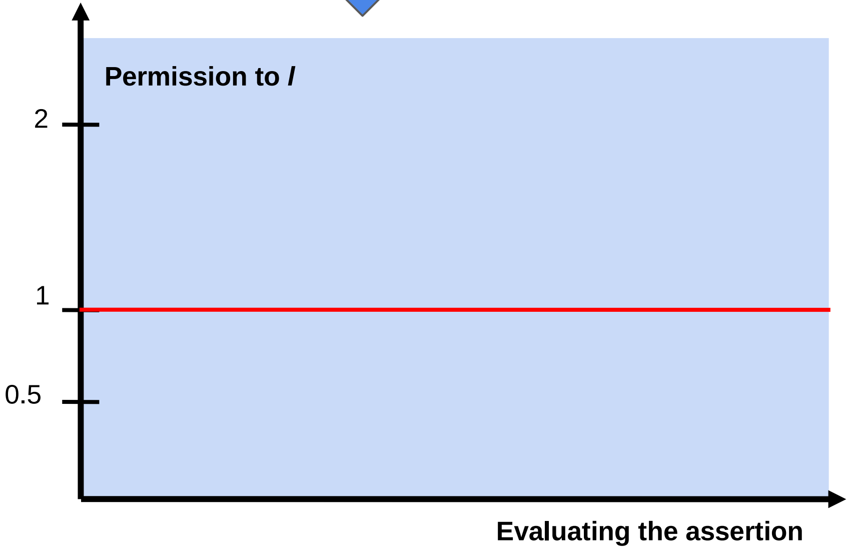
Syntactic

$$0.5 \cdot (l \mapsto v) * 0.5 \cdot (l \mapsto v)$$

Syntactic

Syntactic





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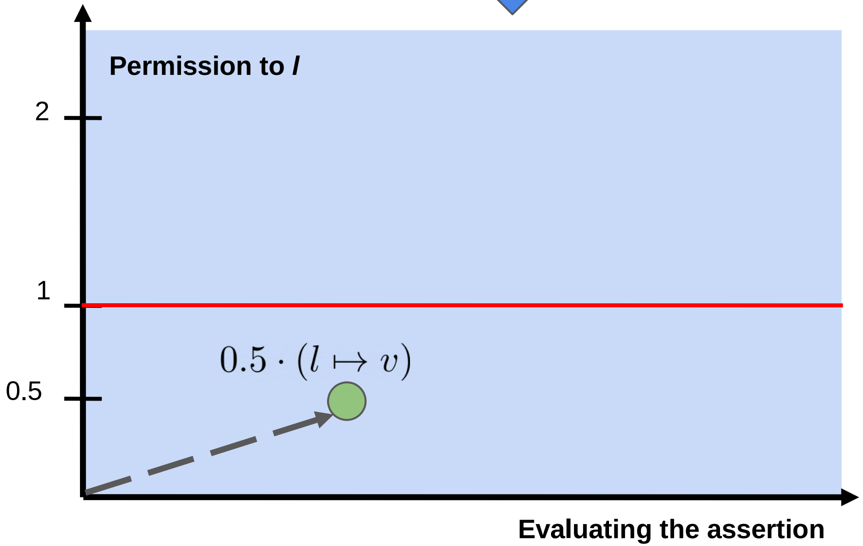

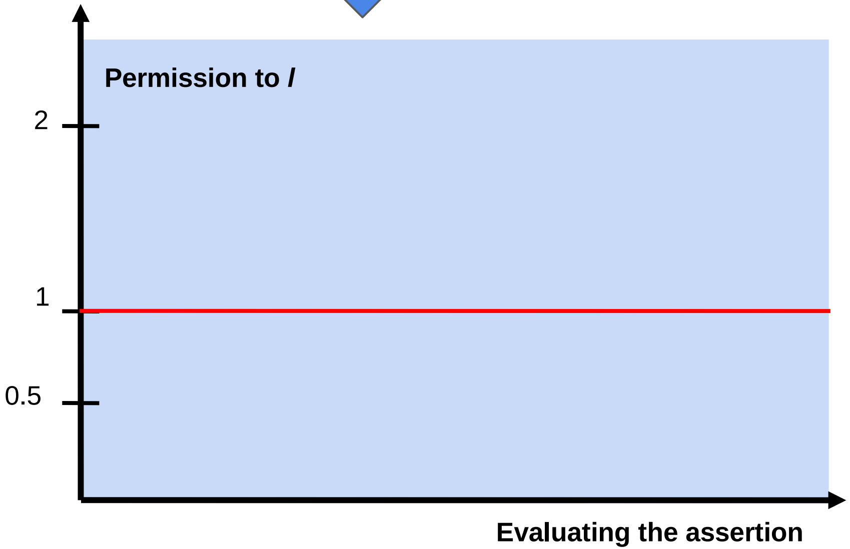

Unbounded separation logic: Intuition

Semantic

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Syntactic

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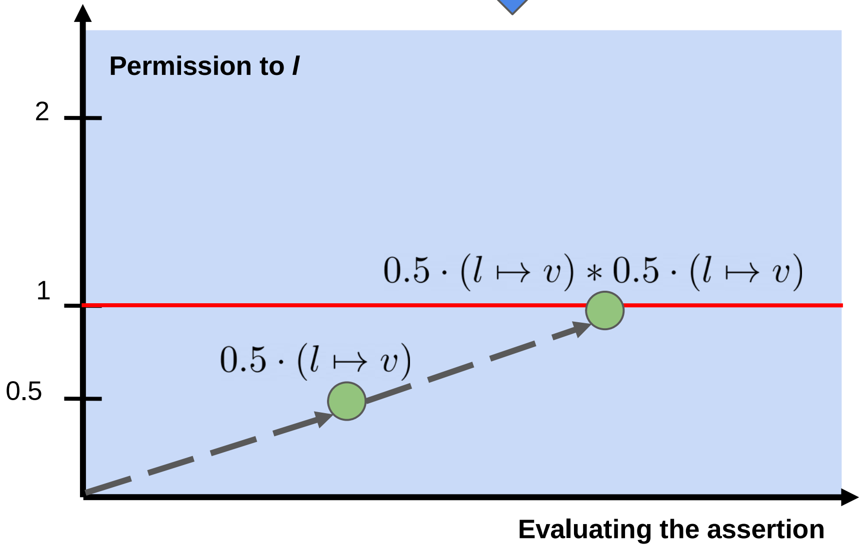
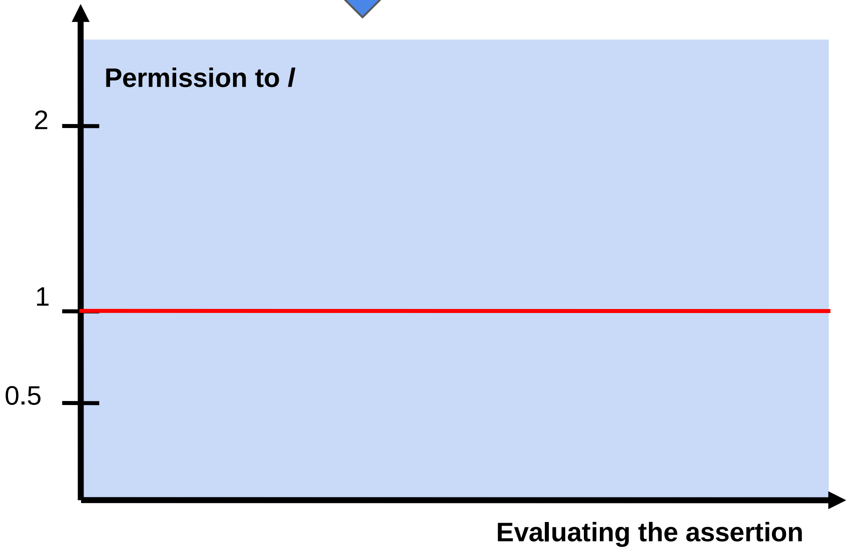
Syntactic

Syntactic





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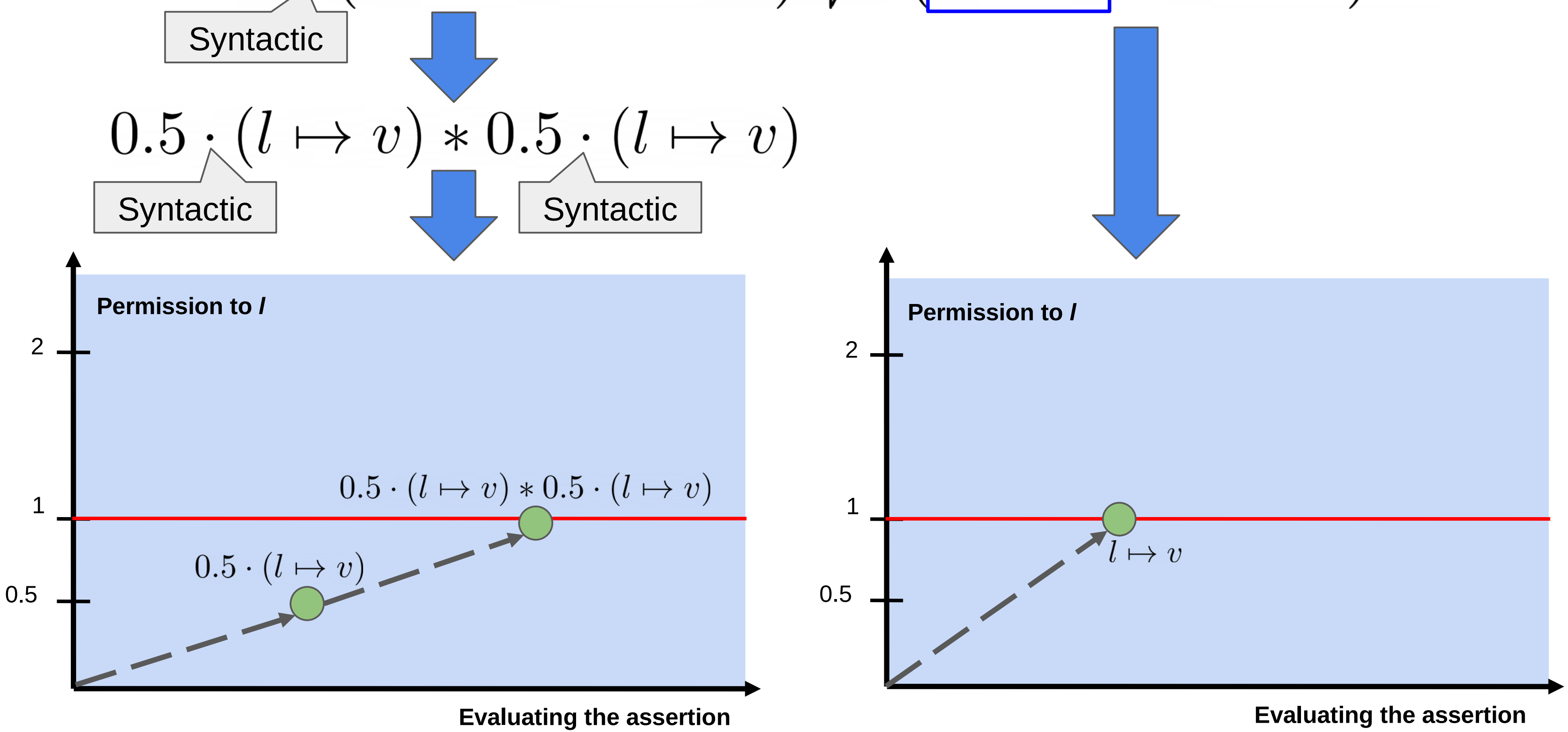

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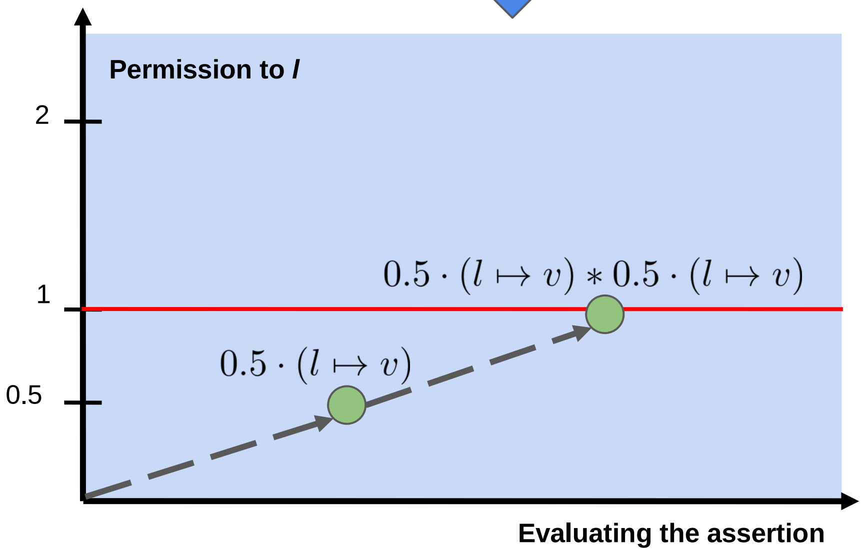

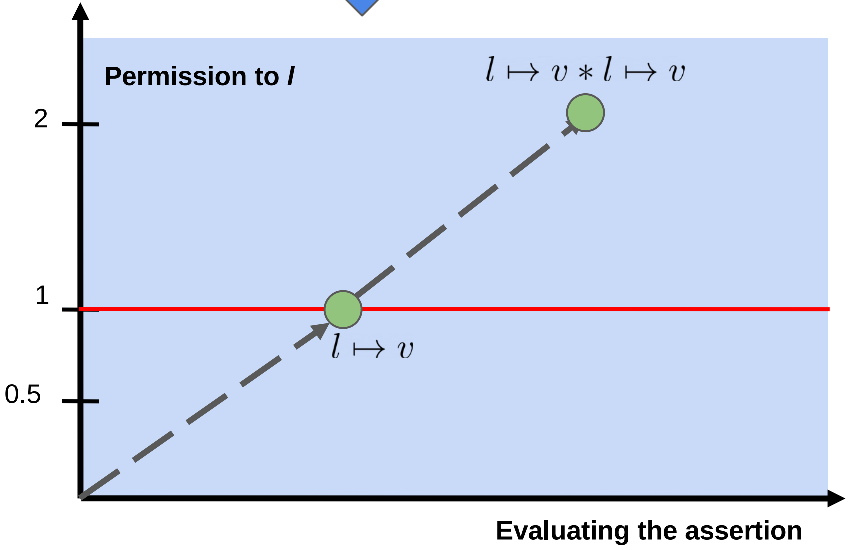

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Syntactic

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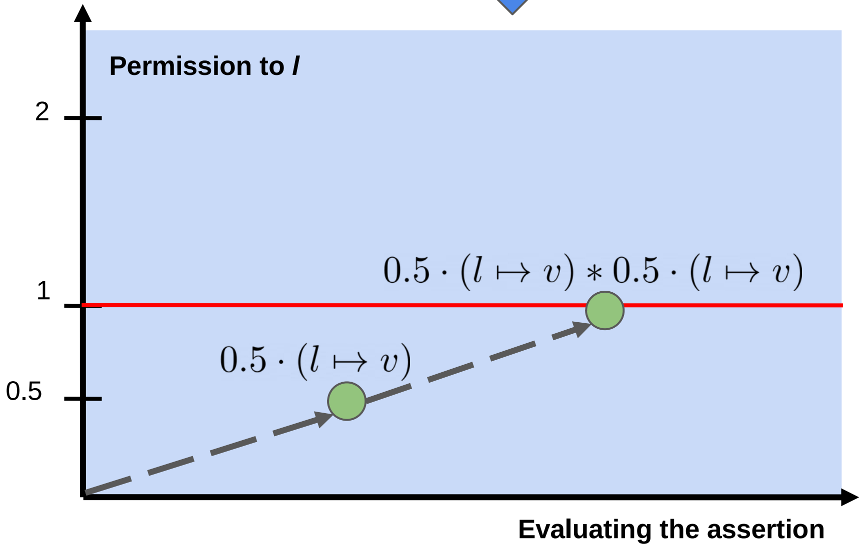
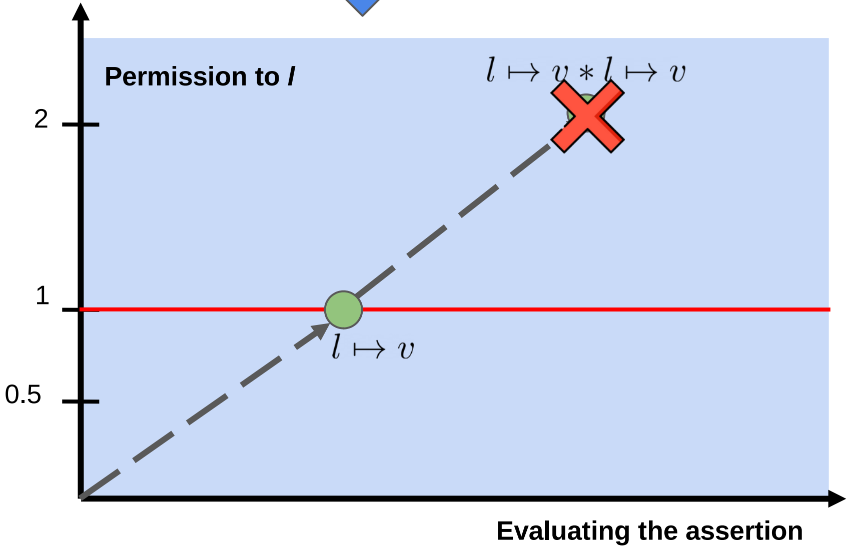
Syntactic





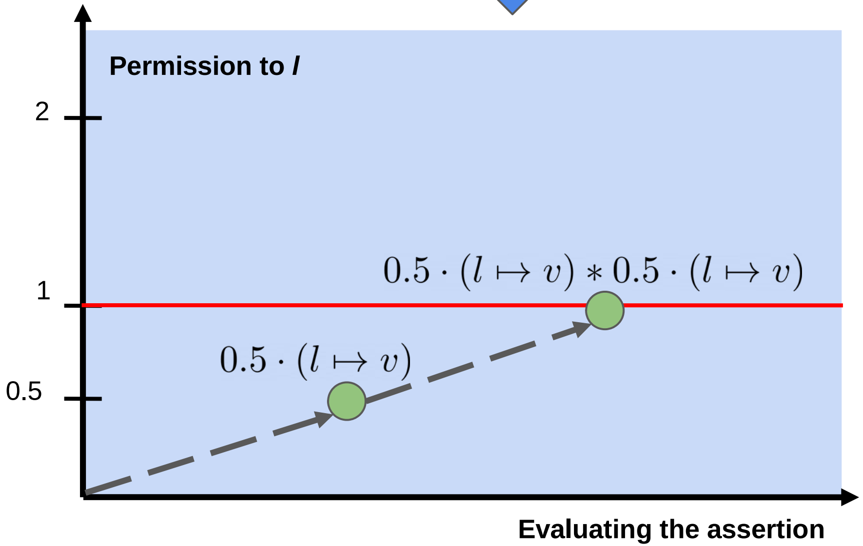
Unbounded separation logic: Intuition

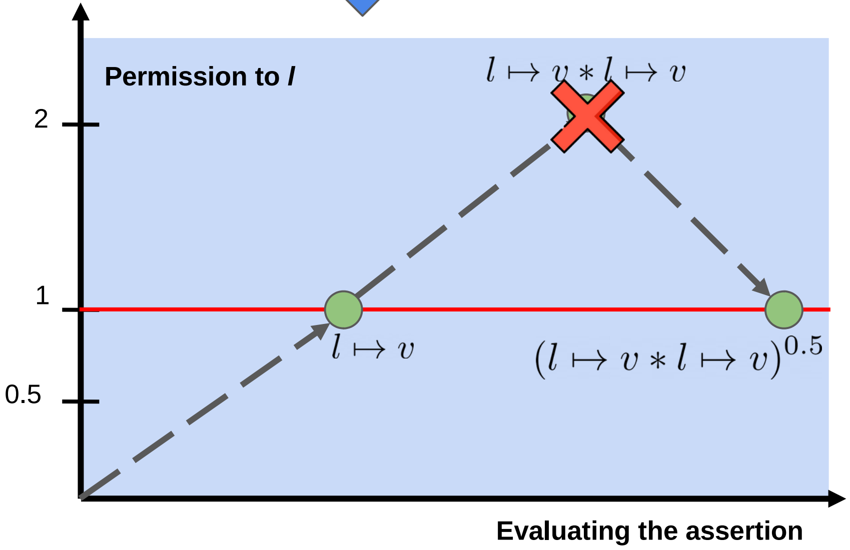
$$0.5 \cdot (l \mapsto v * l \mapsto v) \not\models (l \mapsto v * l \mapsto v)^{0.5}$$

Syntactic | Semantic

$$0.5 \cdot (l \mapsto v) * 0.5 \cdot (l \mapsto v)$$

Syntactic | Syntactic





Unbounded separation logic: Intuition

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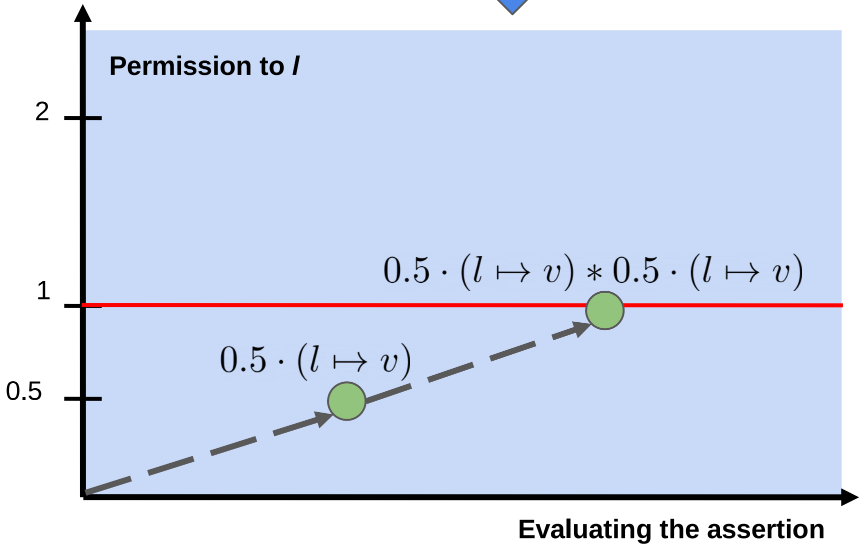

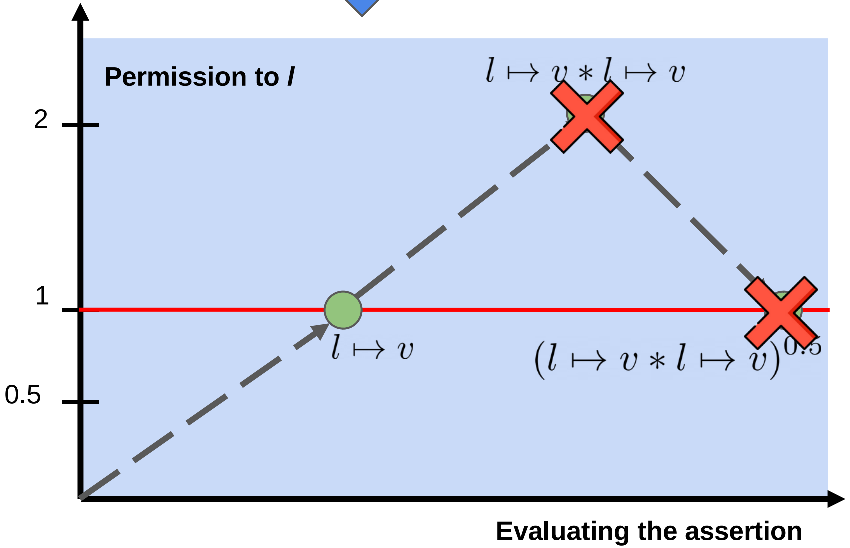

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Syntactic

Semantic

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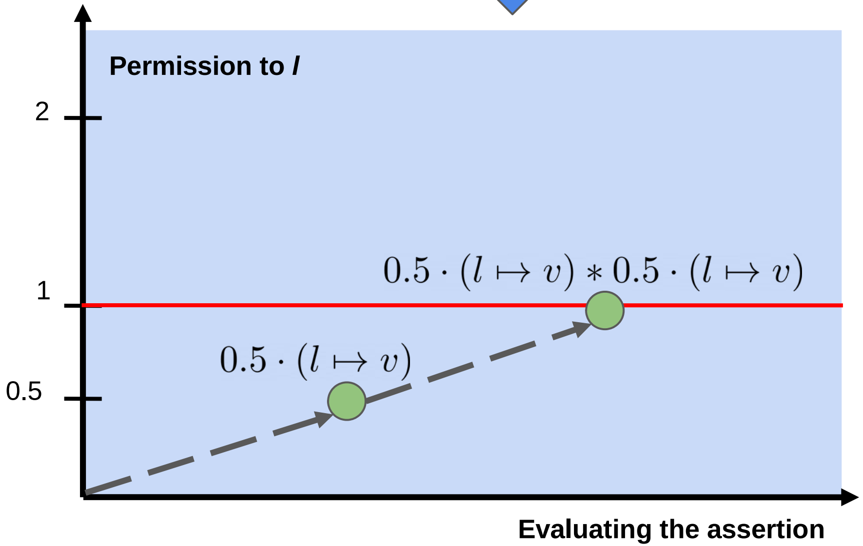
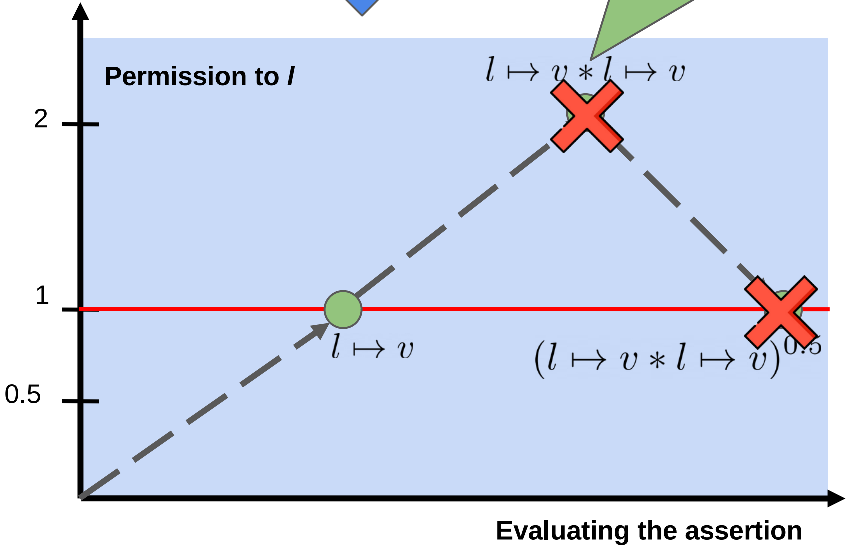
Syntactic

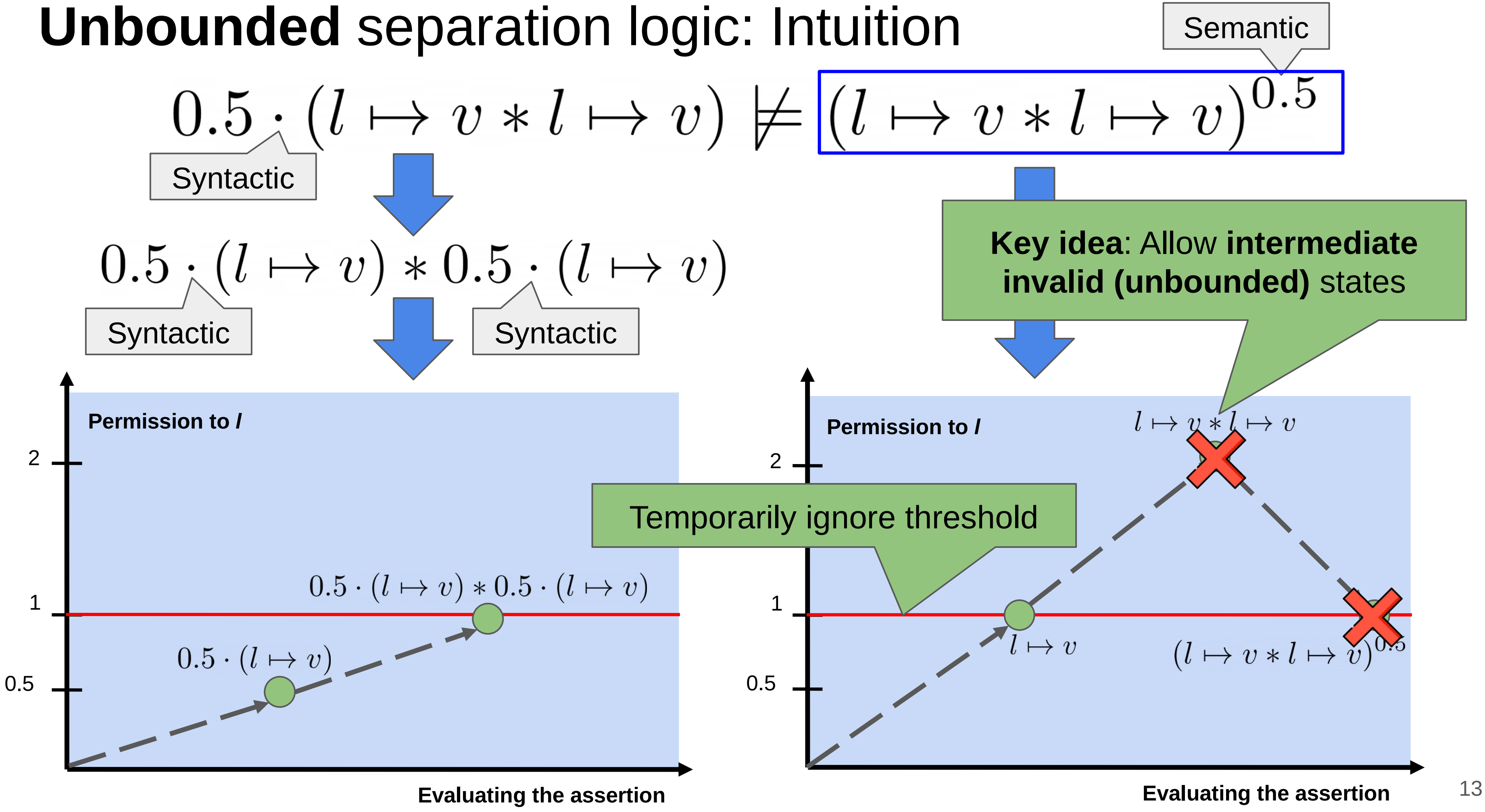
Syntactic

Key idea: Allow **intermediate invalid (unbounded)** states





Unbounded separation logic: Intuition



Unbounded separation logic (simplified)

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1. *Temporarily* allow **unbounded** states in the assertion logic

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Unbounded separation logic (simplified)

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$$BoundedState \triangleq Locations \rightharpoonup Value \times (\mathbb{Q} \cap (0, 1])$$

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Unbounded separation logic (simplified)

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Unbounded separation logic (simplified)

1. *Temporarily* allow **unbounded** states in the assertion logic

$$BoundedState \triangleq Locations \rightharpoonup Value \times \boxed{(\mathbb{Q} \cap (0,1])}$$

$$State \triangleq Locations \rightharpoonup Value \times \boxed{\mathbb{Q}^+}$$

2. Reimpose boundedness at statement boundaries

Unbounded separation logic (simplified)

1. *Temporarily* allow **unbounded** states in the assertion logic

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2. Reimpose boundedness at statement boundaries

$$\{P\}C\{Q\} \iff (\forall h.\, h \models P \Rightarrow \ldots)$$

Unbounded separation logic (simplified)

1. *Temporarily* allow **unbounded** states in the assertion logic

$$BoundedState \triangleq Locations \rightharpoonup Value \times \boxed{(\mathbb{Q} \cap (0, 1])}$$

$$\Downarrow$$

$$State \triangleq Locations \rightharpoonup Value \times \boxed{\mathbb{Q}^+}$$

2. Reimpose boundedness at statement boundaries

$$\{P\}C\{Q\} \iff (\forall h.\, h \models P \Rightarrow \ldots)$$

$$\Downarrow$$

$$\{P\}C\{Q\} \iff (\forall h.\, h \models P \land \boxed{h \in BoundedState} \Rightarrow \ldots)$$

Theoretical foundation for the syntactic multiplication

Theorem: In unbounded separation logic,

$$h \models \pi \cdot A \iff (\exists h_A .\, h_A \models A \land h = \pi \odot h_A)$$

Syntactic

Theoretical foundation for the syntactic multiplication

Theorem: In unbounded separation logic,

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Syntactic

Same definition as the semantic multiplication. The difference is in the **state model** (*bounded* vs. *unbounded*).

What about the frame rule?

$$\frac{\{P\}\ C\ \{Q\} \quad mod(C) \cap fv(R) = \emptyset}{\{P * R\}\ C\ \{Q * R\}}\ (Frame)$$

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$$\frac{\{P\}\ C\ \{Q\} \quad mod(C) \cap fv(R) = \emptyset}{\{P * R\}\ C\ \{Q * R\}}\ (\mathit{Frame})$$

2. Reimpose boundedness at statement boundaries

Theorem: The frame rule holds in unbounded separation logic.

Factorisation and distribution in unbounded separation logic

$$\frac{}{\alpha \cdot (\beta \cdot A) \equiv (\alpha \times \beta) \cdot A}\ (\mathit{DotDot}) \qquad \frac{}{\pi \cdot (\exists x.\, A) \equiv \exists x.\, (\pi \cdot A)}\ (\mathit{DotExists})$$

$$\frac{}{\pi \cdot (A \mathrel{-\!\!*} B) \equiv (\pi \cdot A) \mathrel{-\!\!*} (\pi \cdot B)}\ (\mathit{DotWand}) \qquad \frac{}{\pi \cdot (A \Rightarrow B) \equiv (\pi \cdot A) \Rightarrow (\pi \cdot B)}\ (\mathit{DotImp})$$

$$\frac{}{A \models B \Longleftrightarrow \pi \cdot A \models \pi \cdot B}\ (\mathit{DotPos}) \qquad \frac{}{\pi \cdot (\forall x.\, A) \equiv \forall x.\, (\pi \cdot A)}\ (\mathit{DotForall})$$

$$\frac{}{\pi \cdot (A \wedge B) \equiv (\pi \cdot A) \wedge (\pi \cdot B)}\ (\mathit{DotAnd}) \qquad \frac{}{\pi \cdot (A \vee B) \equiv (\pi \cdot A) \vee (\pi \cdot B)}\ (\mathit{DotOr}) \qquad \frac{}{1 \cdot A \equiv A}\ (\mathit{DotFull})$$

$$\frac{\mathit{pure}(A)}{\pi \cdot A \equiv A}\ (\mathit{DotPure}) \qquad \frac{}{\pi \cdot (A * B) \equiv (\pi \cdot A) * (\pi \cdot B)}\ (\mathit{DotStar}) \qquad \frac{}{(\alpha + \beta) \cdot A \models (\alpha \cdot A) * (\beta \cdot A)}\ (\mathit{Split})$$

Factorisation and distribution in unbounded separation logic

$$\frac{}{\alpha \cdot (\beta \cdot A) \equiv (\alpha \times \beta) \cdot A}\ (DotDot) \qquad \frac{}{\pi \cdot (\exists x.\, A) \equiv \exists x.\, (\pi \cdot A)}\ (DotExists)$$

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Factorisation and distribution in unbounded separation logic

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Factorisation and distribution in unbounded separation logic

The syntactic multiplication can be extended to support fractional magic wands

$$\frac{}{\ldots \cdot (\exists x.\, A) \equiv \exists x.\, (\pi \cdot A)}\ (DotExists)$$

$$\frac{}{\pi \cdot (A \mathrel{-\!\!*} B) \equiv (\pi \cdot A) \mathrel{-\!\!*} (\pi \cdot B)}\ (DotWand) \qquad \frac{}{\pi \cdot (A \Rightarrow B) \equiv (\pi \cdot A) \Rightarrow (\pi \cdot B)}\ (DotImp)$$

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Using fractional resources

```
method processTree(x: Ref) {
  {tree(x)^π}
  if (x != null) {
    {tree(x)^π * x ≠ null}
    {(tree(x)^(π/2) * x ≠ null) * (tree(x)^(π/2) * x ≠ null)}
        {tree(x)^(π/2) * x ≠ null}
        {x.val ↦^(π/2) _ * ... * tree(x_l)^(π/2) * tree(x_r)^(π/2)}
        print(x.val)
        processTree(x.left)
        processTree(x.right)

        {x.val ↦^(π/2) _ * ... * tree(x_l)^(π/2) * tree(x_r)^(π/2)}
        {tree(x)^(π/2)}
    {tree(x)^(π/2) * tree(x)^(π/2)}
  }
  {tree(x)^π}
}
```

1. Split

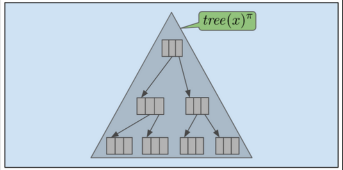
2. Distribute

3. Factorise

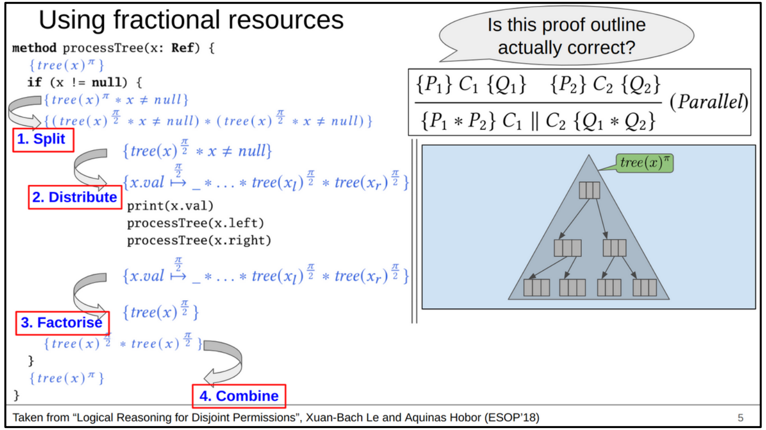
4. Combine

Is this proof outline actually correct?

$$\frac{\{P_1\}\ C_1\ \{Q_1\} \quad \{P_2\}\ C_2\ \{Q_2\}}{\{P_1 * P_2\}\ C_1 \parallel C_2\ \{Q_1 * Q_2\}}\ (Parallel)$$



Taken from "Logical Reasoning for Disjoint Permissions", Xuan-Bach Le and Aquinas Hobor (ESOP'18) 5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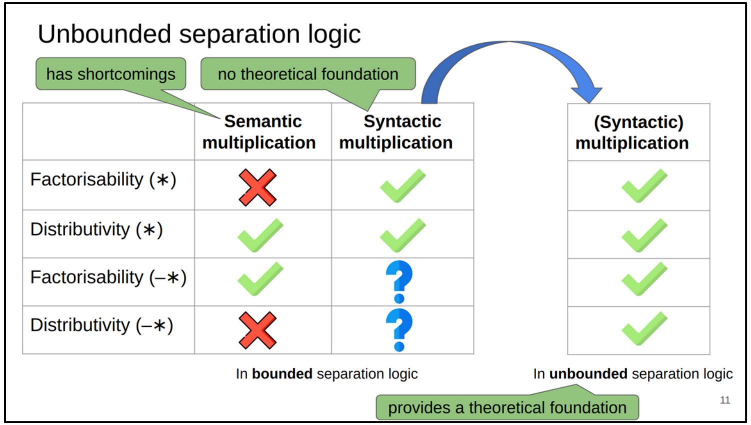
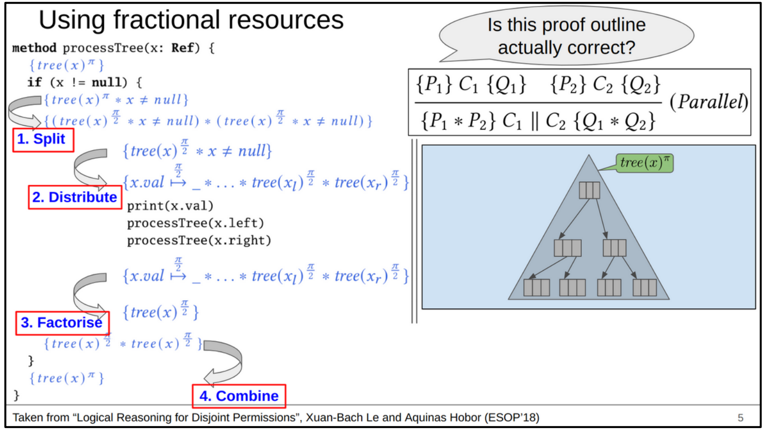
5


Unbounded separation logic
has shortcomings
no theoretical foundation
Semantic multiplication
Syntactic multiplication
(Syntactic) multiplication
Factorisability (∗)
Distributivity (∗)
Factorisability (−∗)
Distributivity (−∗)
In bounded separation logic
In unbounded separation logic
provides a theoretical foundation
11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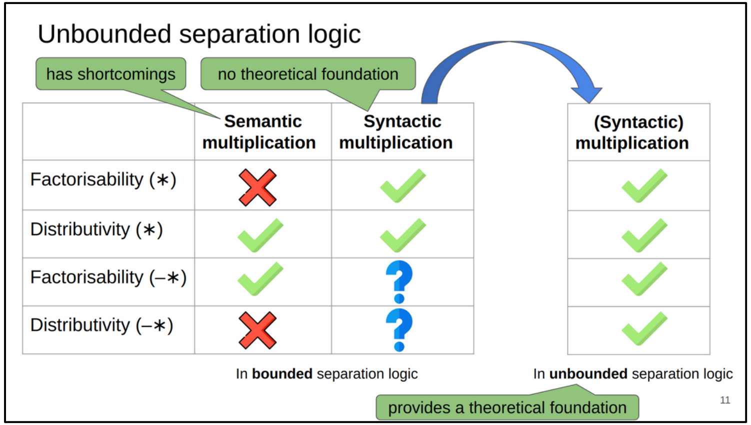
Taken from "Logical Reasoning for Disjoint Permissions", Xuan-Bach Le and Aquinas Hobor (ESOP'18)
5


Unbounded separation logic
has shortcomings
no theoretical foundation
Semantic multiplication
Syntactic multiplication
(Syntactic) multiplication
Factorisability (∗)
Distributivity (∗)
Factorisability (−∗)
Distributivity (−∗)
In bounded separation logic
In unbounded separation logic
provides a theoretical foundation
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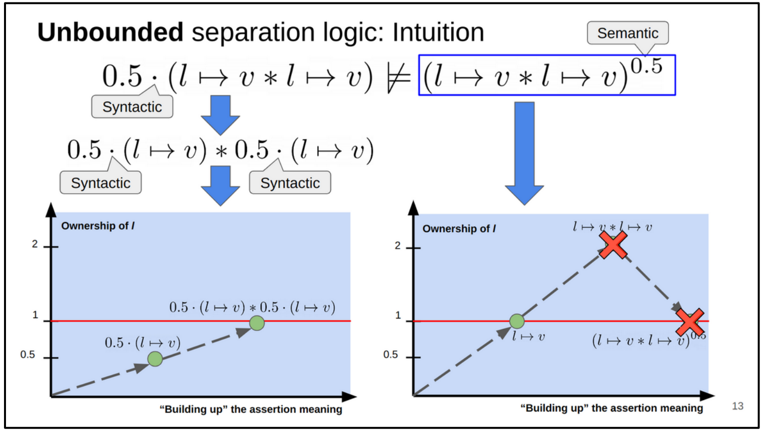
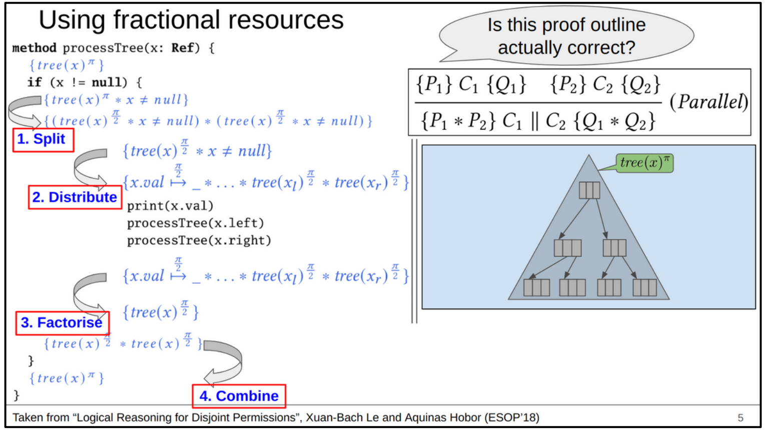
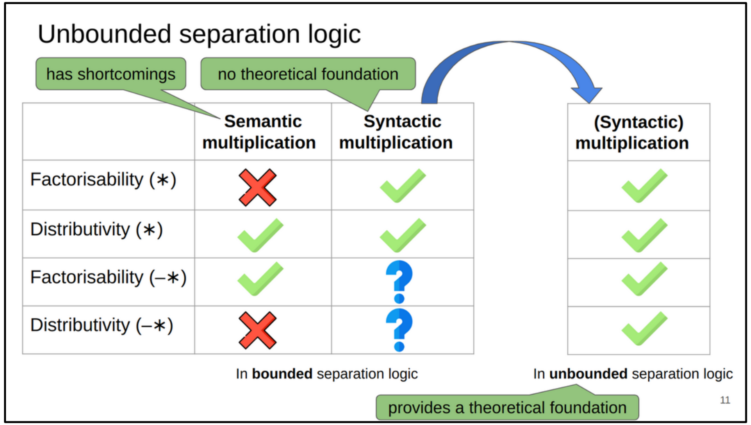
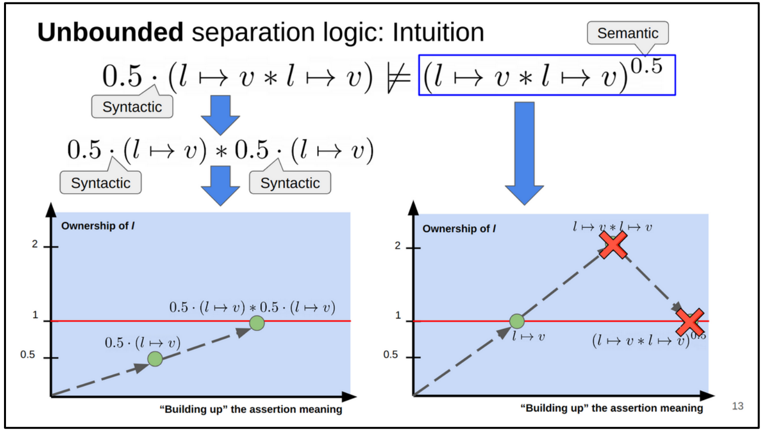

Unbounded separation logic: Intuition
Semantic
Syntactic
Syntactic
Syntactic
Ownership of l
"Building up" the assertion meaning
Ownership of l
"Building up" the assertion meaning
13



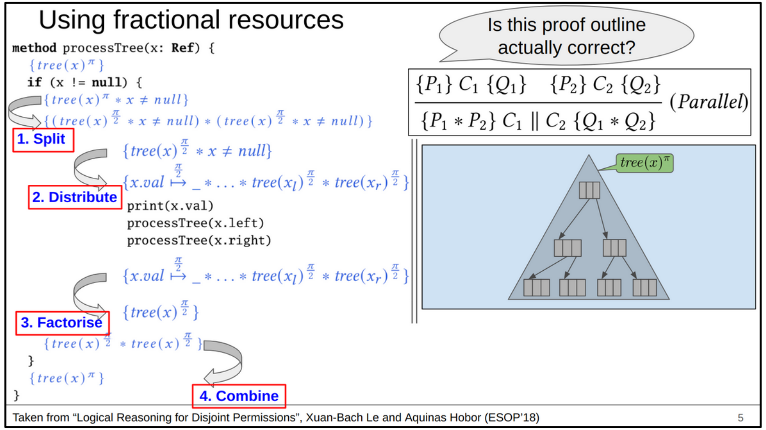


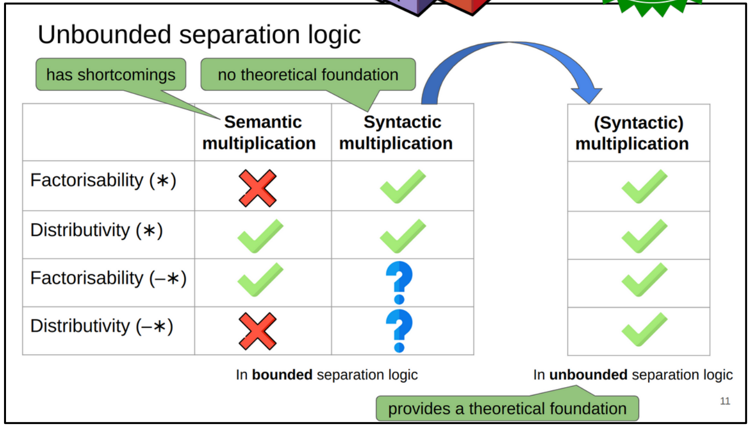


More in the paper:

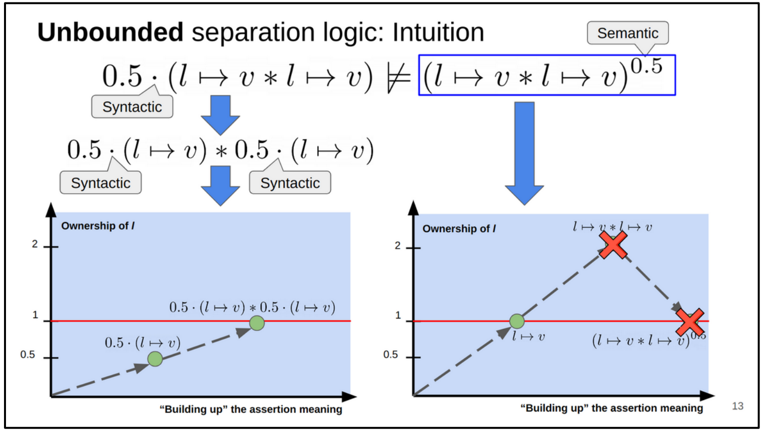
- combinability (step 4)
- reasoning principles for (co)inductive predicates
- unbounded separation logic as a formal foundation for automatic verifiers

Thank you for your attention!





	Semantic multiplication	Syntactic multiplication	(Syntactic) multiplication
Factorisability (∗)	✗	✓	✓
Distributivity (∗)	✓	✓	✓
Factorisability (−∗)	✓	?	✓
Distributivity (−∗)	✗	?	✓
	In **bounded** separation logic		In **unbounded** separation logic



More in the paper:

- ❖ combinability (step 4)
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