Hyper Hoare Logic

(Dis-)Proving Program Hyperproperties

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Hyperproperties

Hyperproperty $\triangleq$ property of a **set** of executions

	Type	Type of negation
Non-interference	$\forall\forall$	$\exists\exists$
Determinism	$\forall\forall$	$\exists\exists$
Monotonicity	$\forall\forall$	$\exists\exists$
Transitivity	$\forall\forall\forall$	$\exists\exists\exists$
Functional correctness	$\forall$	$\exists$
Reachability	$\exists$	$\forall$
Generalized non-interference	$\forall\forall\exists$	$\exists\exists\forall$
Existence of a minimum	$\exists\forall$	$\forall\exists$

Hyperproperties

Hyperproperty ≜ property of a **set** of executions

For any two executions, if they have the same public inputs, they must have the same public outputs

	e	Type of negation
Non-interference	∀∀	∃∃
Determinism	∀∀	∃∃
Monotonicity	∀∀	∃∃
Transitivity	∀∀∀	∃∃∃
Functional correctness	∀	∃
Reachability	∃	∀
Generalized non-interference	∀∀∃	∃∃∀
Existence of a minimum	∃∀	∀∃

Hyperproperties

Hyperproperty ≜ property of a **set** of executions

	Type	[illegible]
Non-interference	∀∀	∃∃
Determinism	∀∀	∃∃
Monotonicity	∀∀	∃∃
Transitivity	∀∀∀	∃∃∃
Functional correctness	∀	∃
Reachability	∃	∀
Generalized non-interference	∀∀∃	∃∃∀
Existence of a minimum	∃∀	∀∃

2-safety hyperproperty

Hyperproperties

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Non-interference	∀∀	∃∃
Determinism	∀∀	∃∃
Monotonicity	∀∀	∃∃
Transitivity	∀∀∀	∃∃∃
Functional correctness		
Reachability	∃	∀
Generalized non-interference	∀∀∃	∃∃∀
Existence of a minimum	∃∀	∀∃

Intuitively, "∀x. ∀y. x ≥ y ⇒ **f(x)** ≥ **f(y)**"

Hyperproperties

Hyperproperty ≜ property of a **set** of executions

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Non-interference	∀∀	∃∃
Determinism	∀∀	∃∃
Monotonicity	∀∀	∃∃
Transitivity	∀∀∀	∃∃∃
Functional correctness	∀	∃
Reachability	∃	∀
Generalized non-interference	∀∀∃	∃∃∀
Existence of a minimum	∃∀	∀∃

Intuitively, "∀x. ∀y. ∀z. **f(x, y)** ∧ **f(y, z)** ⇒ **f(x, z)**"

Hyperproperties

Hyperproperty ≜ property of a **set** of executions

Intuitively, "$\exists x.\ \exists y.\ x \geq y \ \wedge\ \mathbf{f(x)} < \mathbf{f(y)}$"

	Type	Type ... on
Non-interference	$\forall\forall$	$\exists\exists$
Determinism	$\forall\forall$	$\exists\exists$
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Hyperproperties

Hyperproperty $\triangleq$ property of a **set** of executions

Goal

Develop the first program logic that can handle all these different types of hyperproperties.

Reachability	$\exists$	$\forall$
Generalized non-interference	$\forall\forall\exists$	$\exists\exists\forall$
Existence of a minimum	$\exists\forall$	$\forall\exists$

Main contribution

Hyper Hoare Logic, a program logic
that allows to (dis-)prove arbitrary program hyperproperties,
including $\forall^*\exists^*$- and $\exists^*\forall^*$-hyperproperties.

Contributions

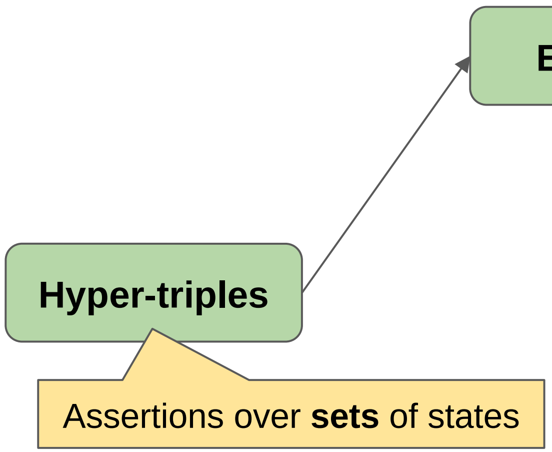
Hyper-triples

Contributions

Hyper-triples

Assertions over **sets** of states

Contributions



Contributions

Properties over sets of **terminating** executions (restricted to initial and final states)

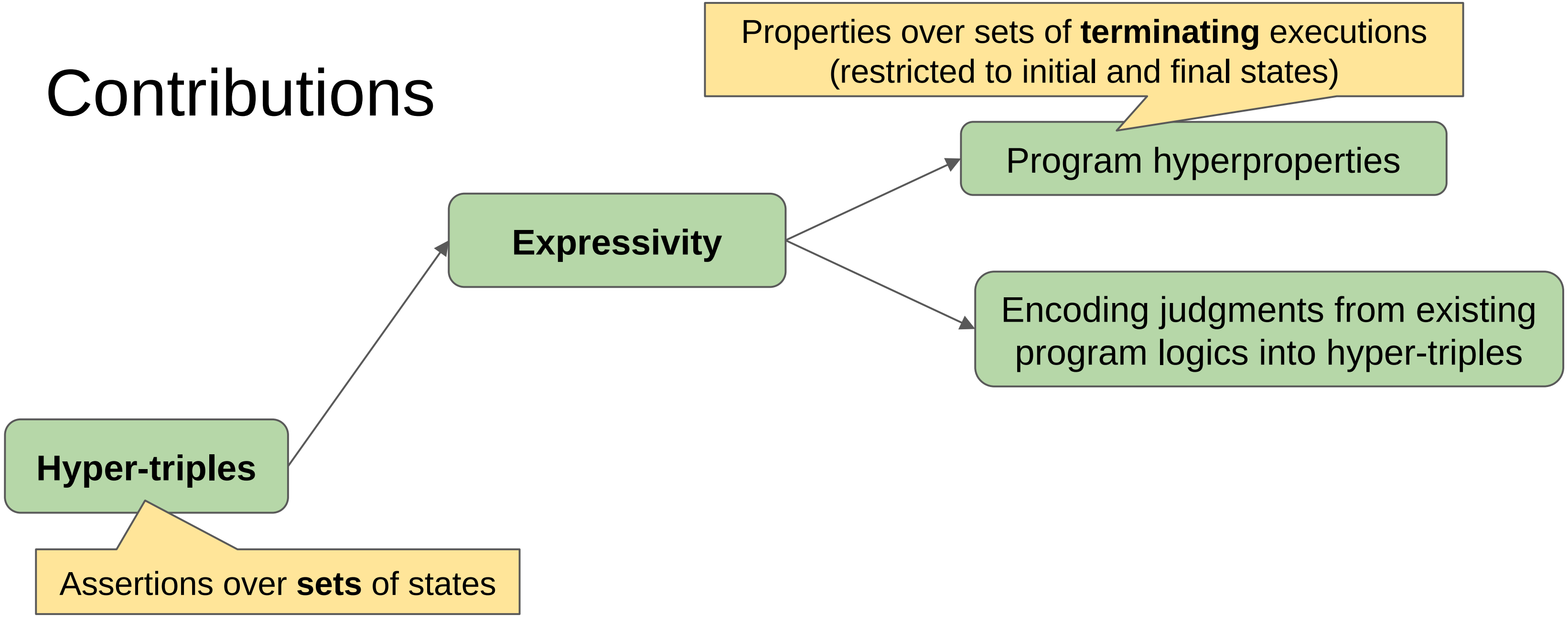
Program hyperproperties

Expressivity

Hyper-triples

Assertions over **sets** of states

Contributions



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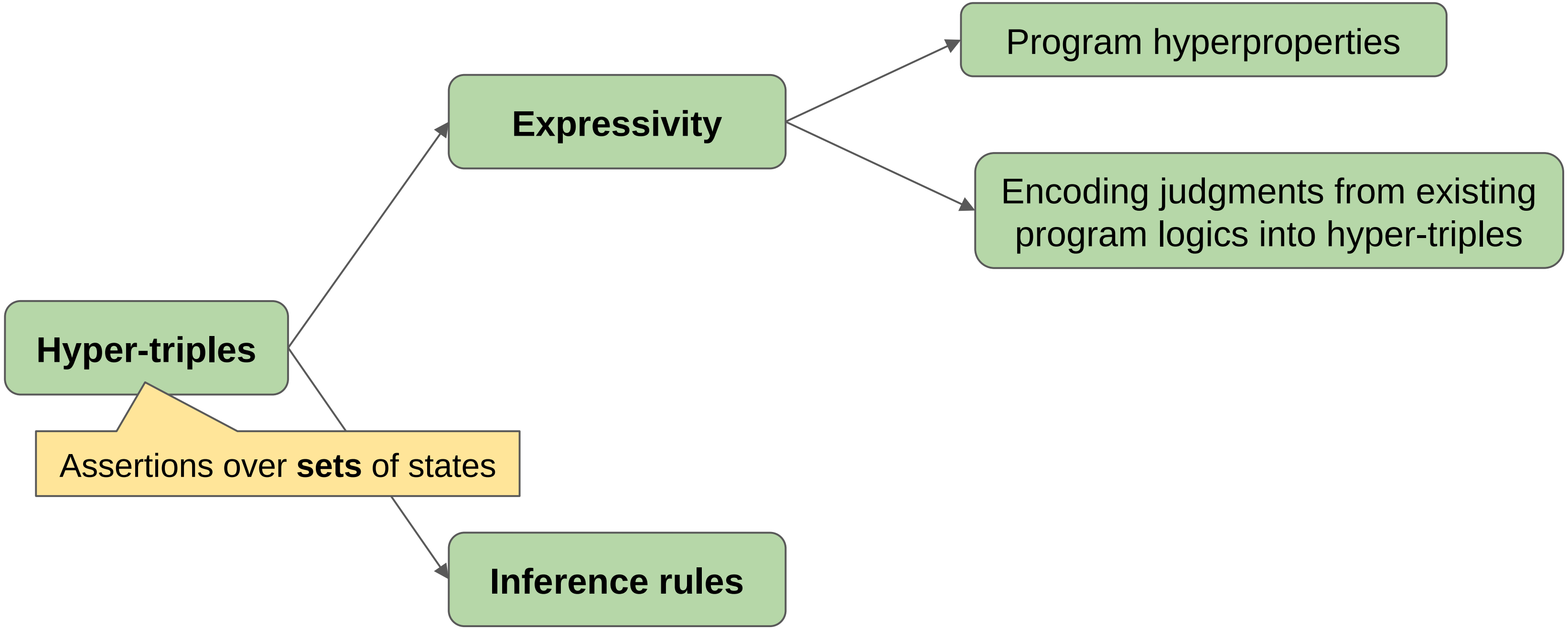

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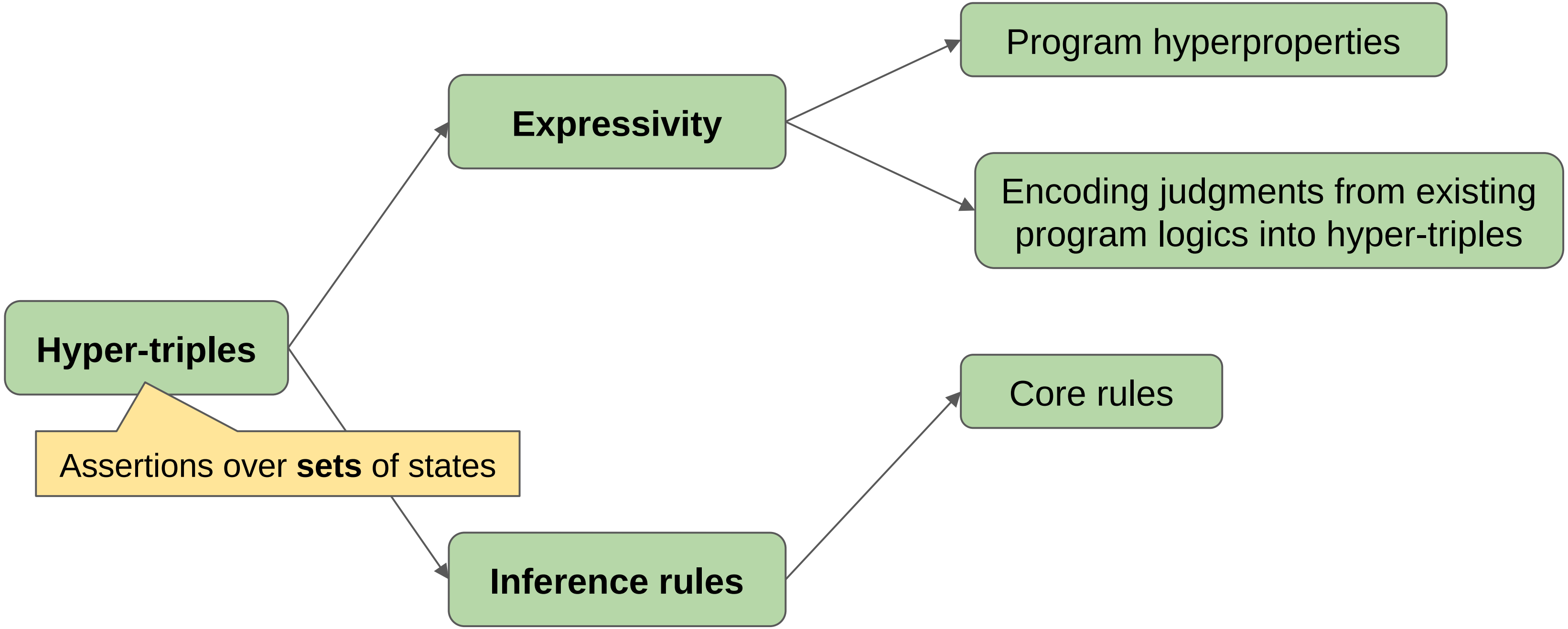

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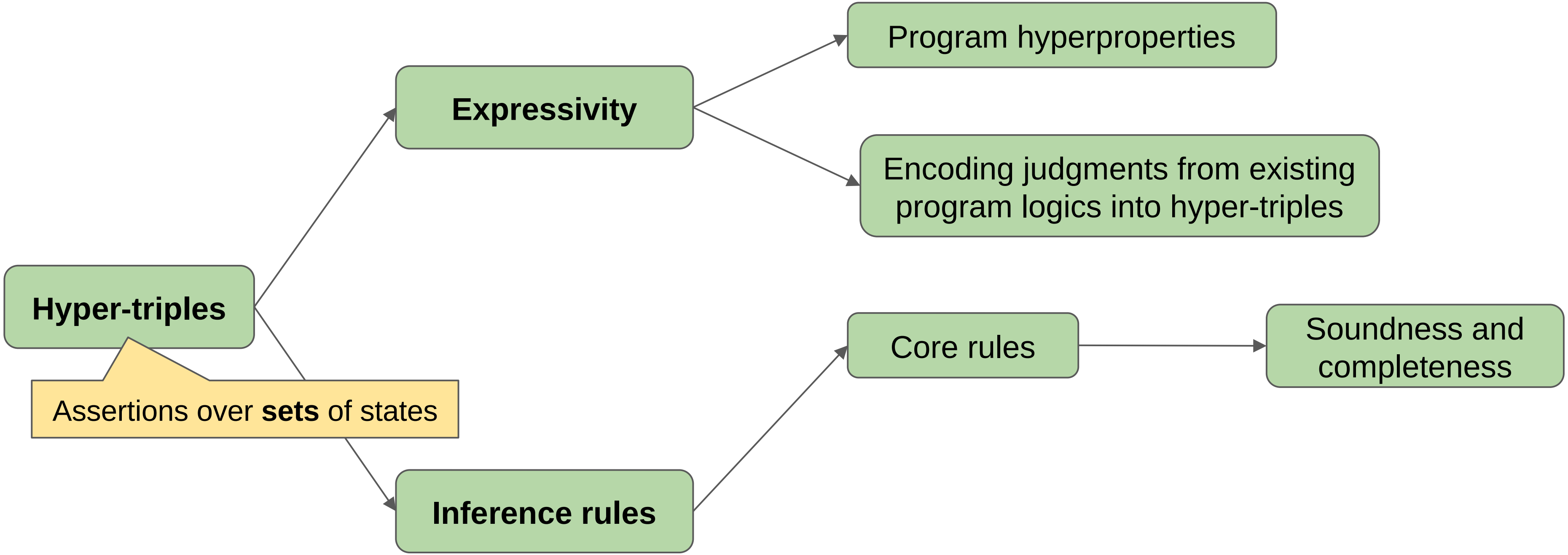

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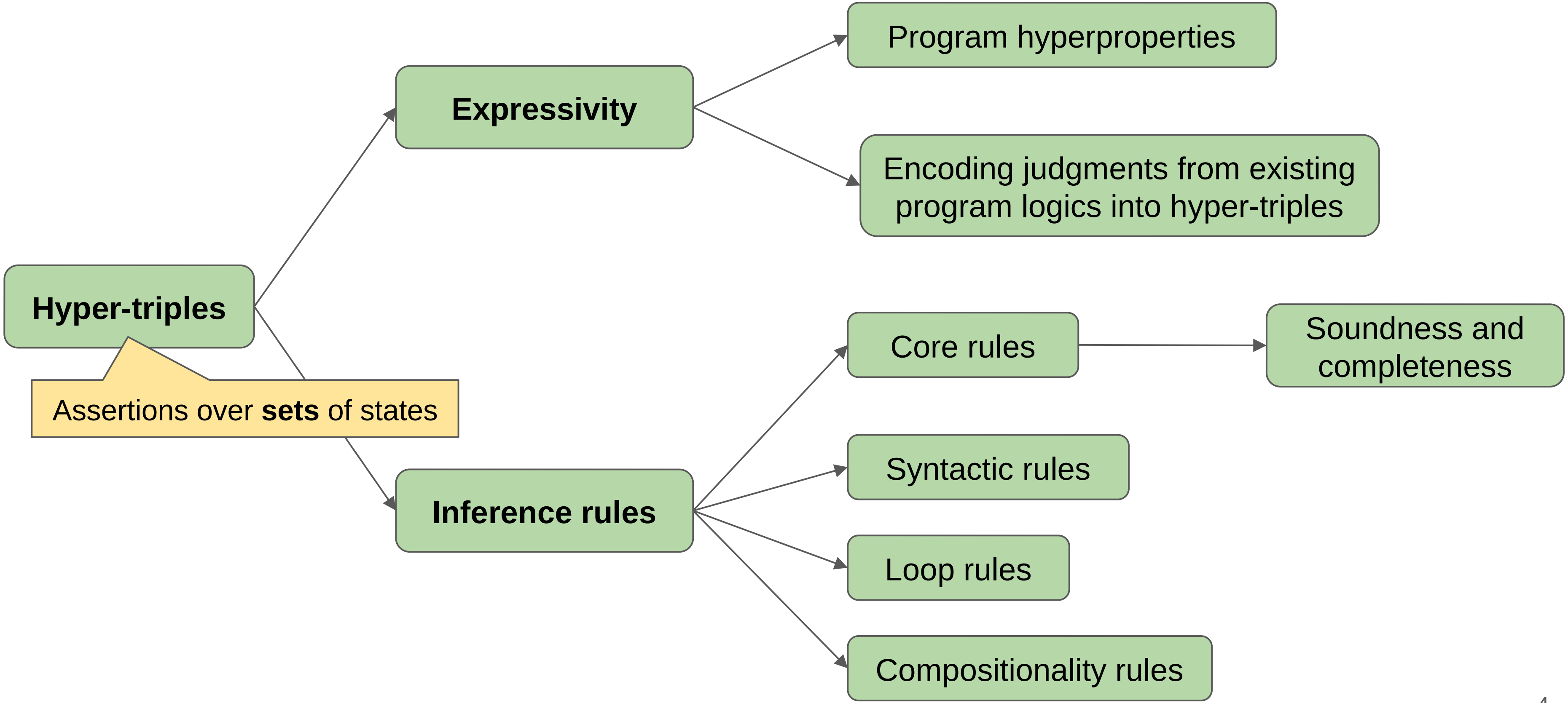

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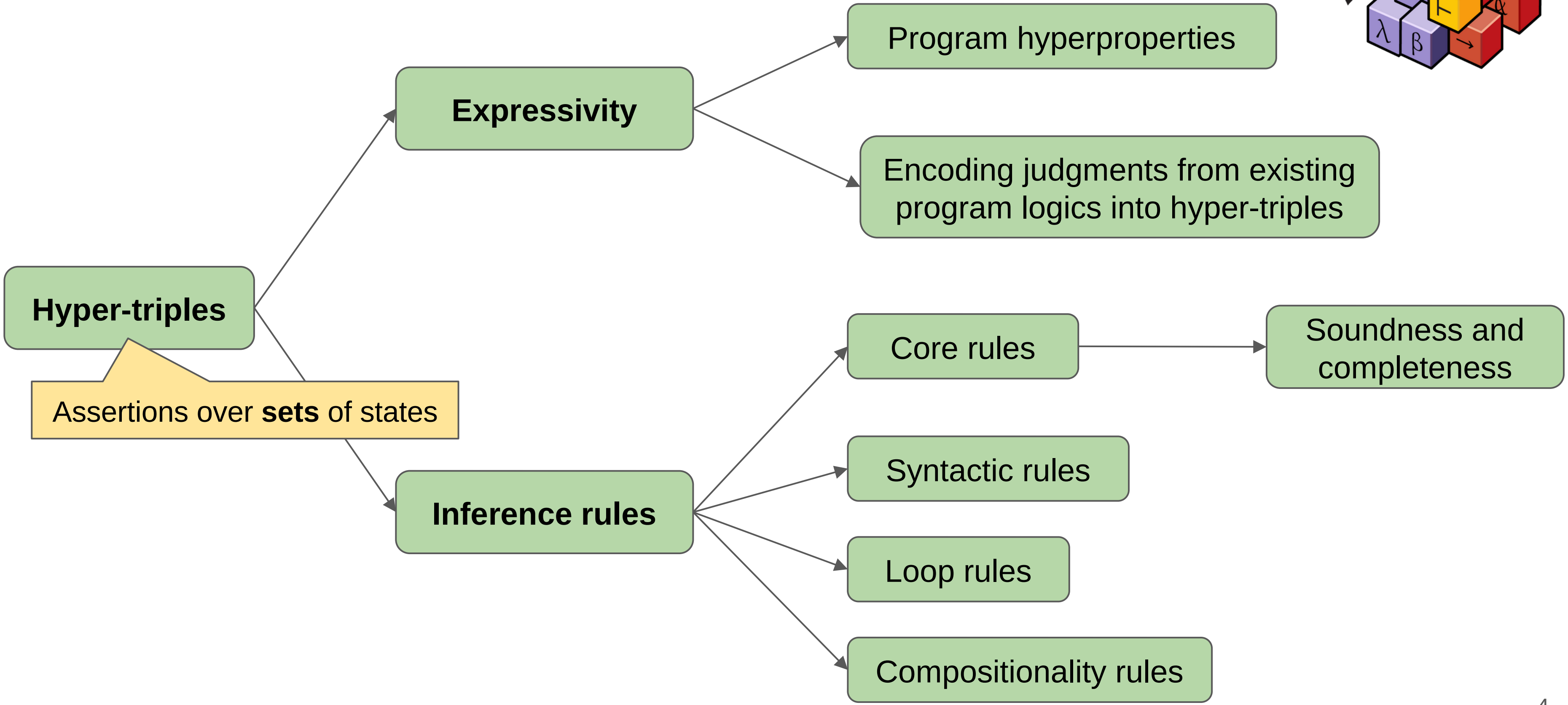

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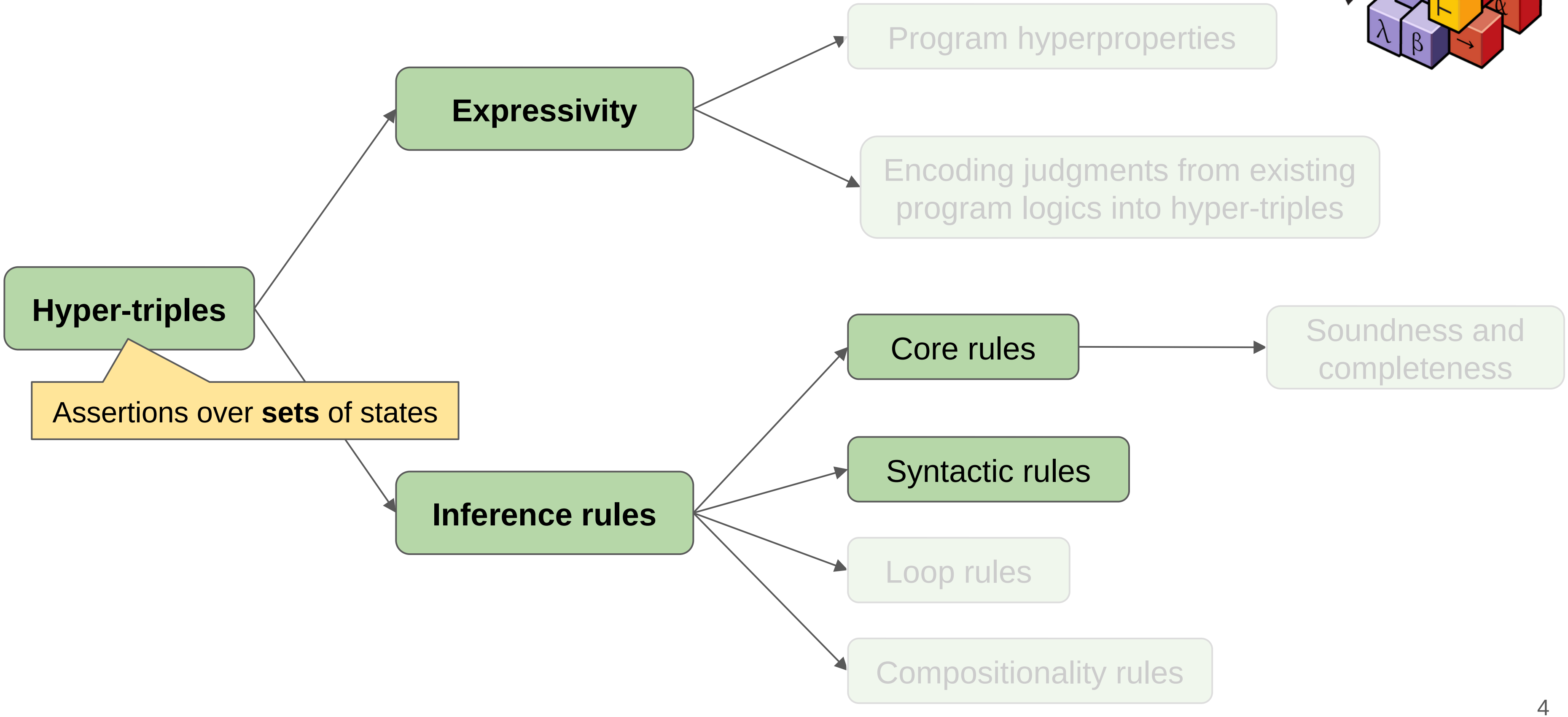

Hyper-Triples: Tracking **Sets** of States

$$\models \{P\}\ C\ \{Q\}$$

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Hyper-assertions
(predicates on **sets** of states)

Hyper-Triples: Tracking **Sets** of States

$$\models \{P\}\ C\ \{Q\} \triangleq (\ \forall S.\ P(S) \Rightarrow Q(\text{sem}(C, S))\ )$$

Hyper-assertions
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Hyper-Triples: Tracking **Sets** of States

Set of initial states

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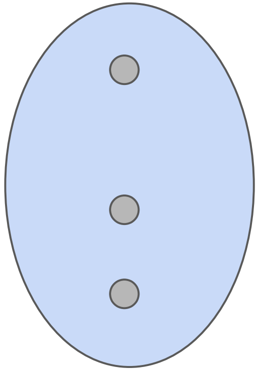

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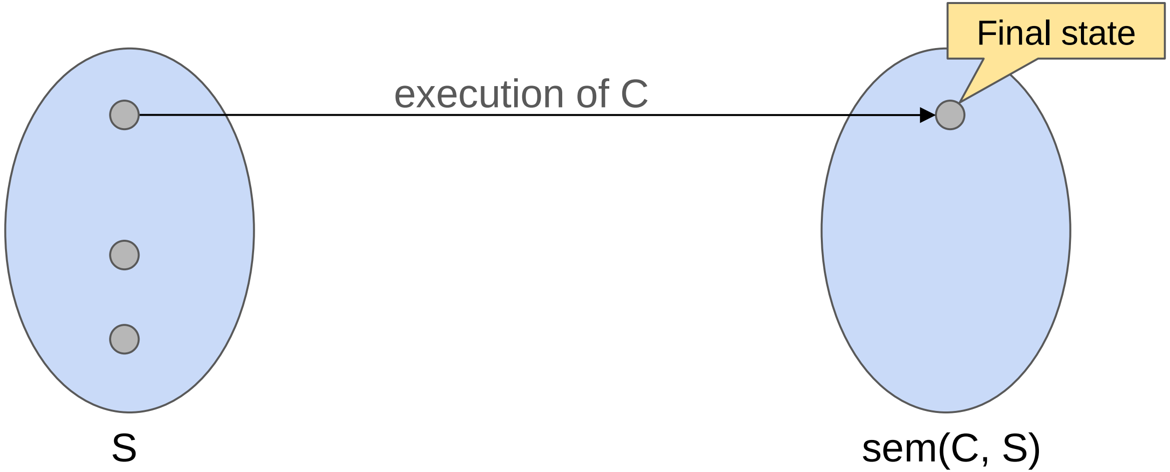

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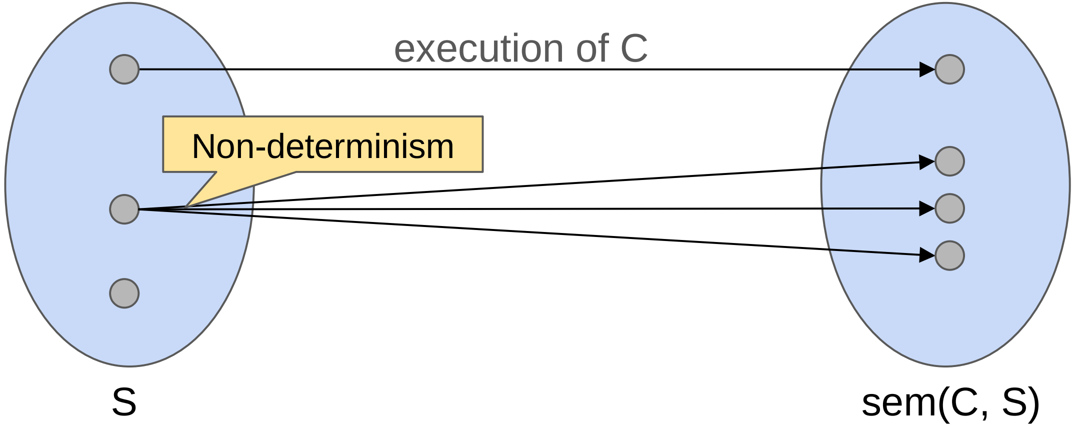

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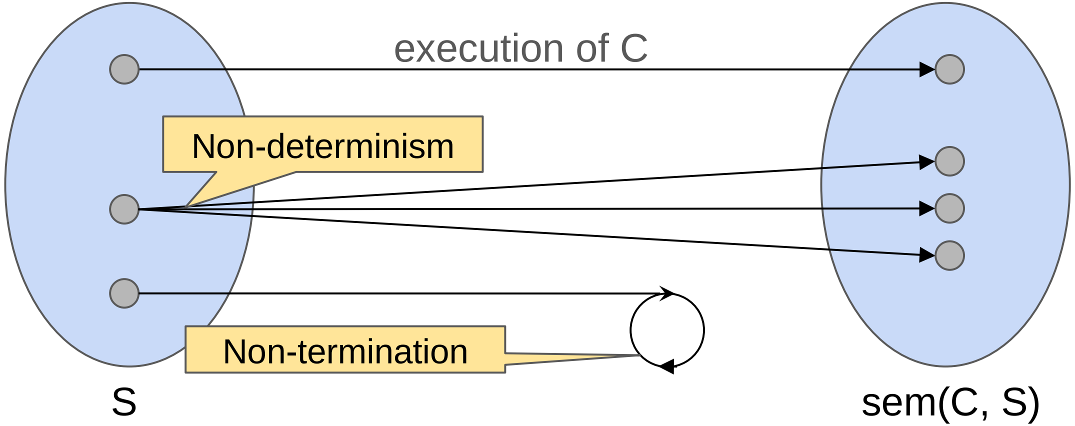

Examples

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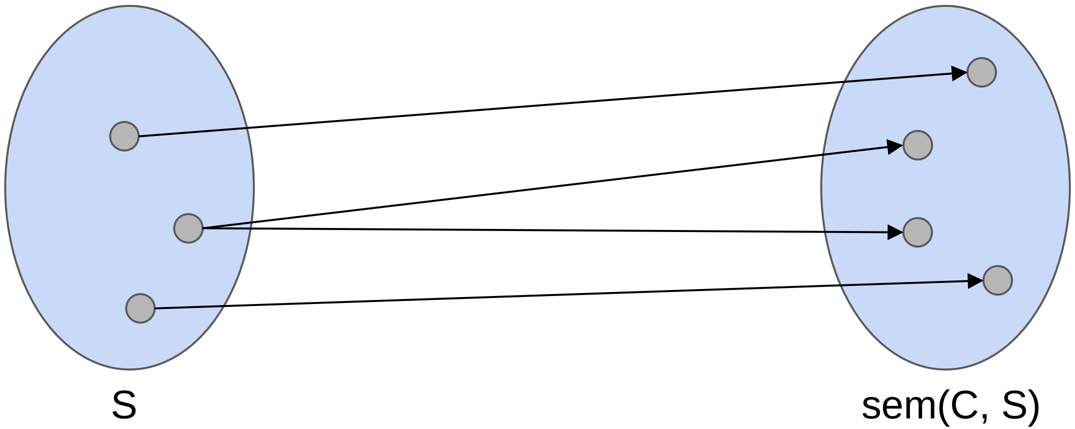
Non-interference (public input x, public output y)



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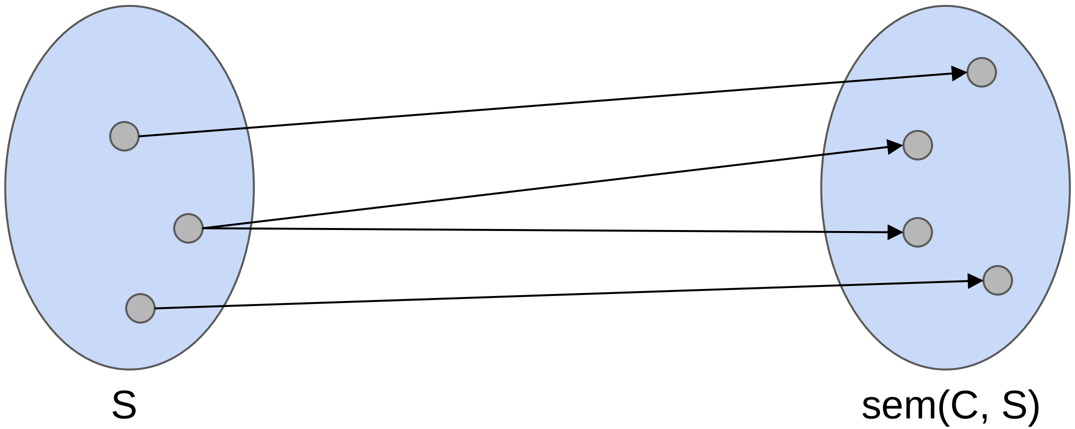
Non-interference (public input x, public output y)

$$\{\forall\langle\sigma_1\rangle.\ \forall\langle\sigma_2\rangle.\ \sigma_1(\mathrm{x})=\sigma_2(\mathrm{x})\ \}\ C\ \{\forall\langle\sigma_1\rangle.\ \forall\langle\sigma_2\rangle.\ \sigma_1(\mathrm{y})=\sigma_2(\mathrm{y})\ \}$$



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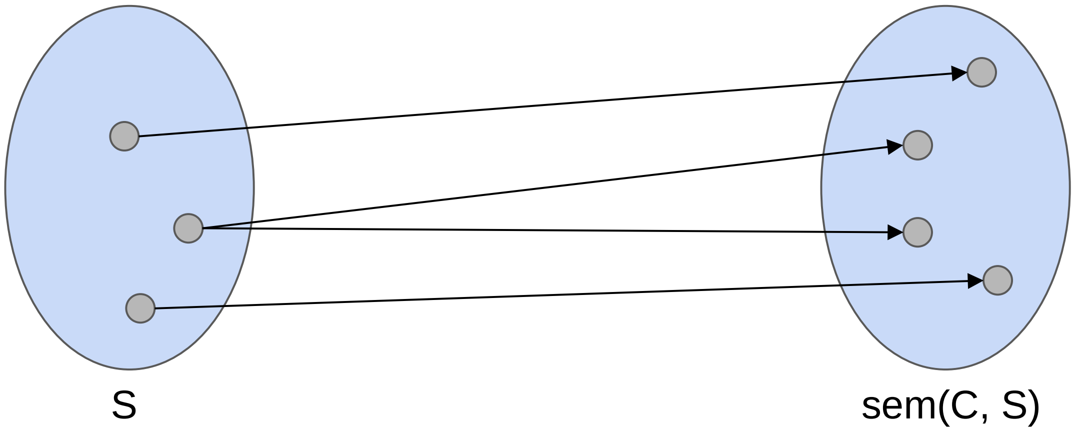

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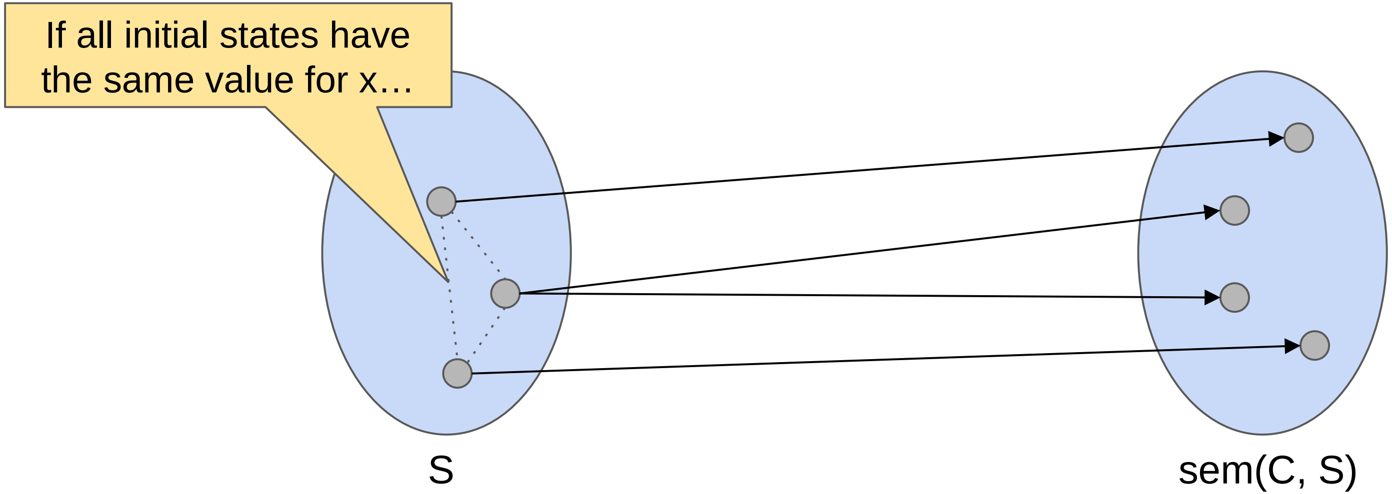

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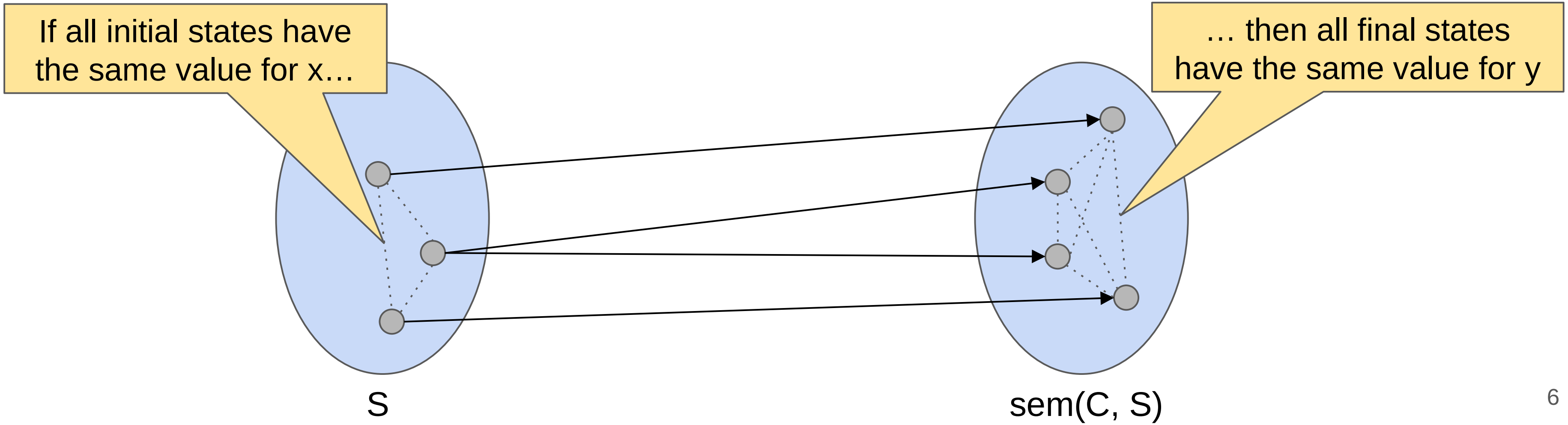

Examples

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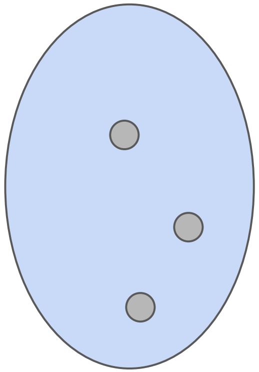
Monotonicity (input x, output y)



S

Examples

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Monotonicity (input x, output y)

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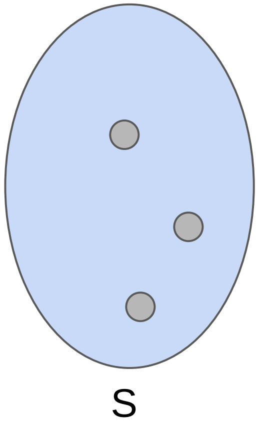

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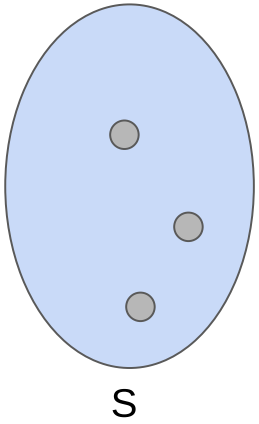
Only satisfiable by the empty set



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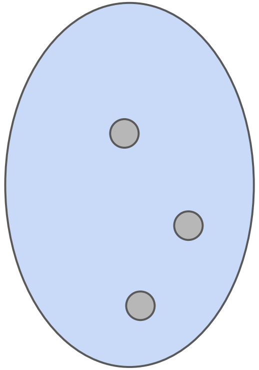

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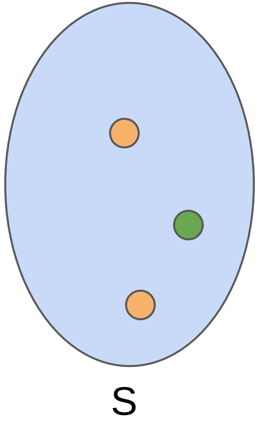

Examples

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Monotonicity (input x, output y)

logical tag to identify executions

$$\cancel{\{\ \forall\langle\sigma_1\rangle.\ \forall\langle\sigma_2\rangle.\ \sigma_1(x) > \sigma_2(x)\ \}\ C\ \{\ \forall\langle\sigma_1\rangle.\ \forall\langle\sigma_2\rangle.\ \sigma_1(y) > \sigma_2(y)\ \}}$$

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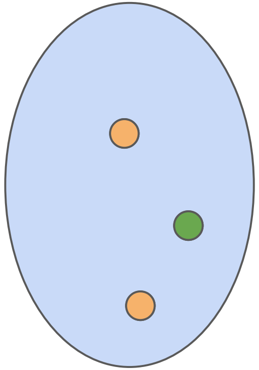

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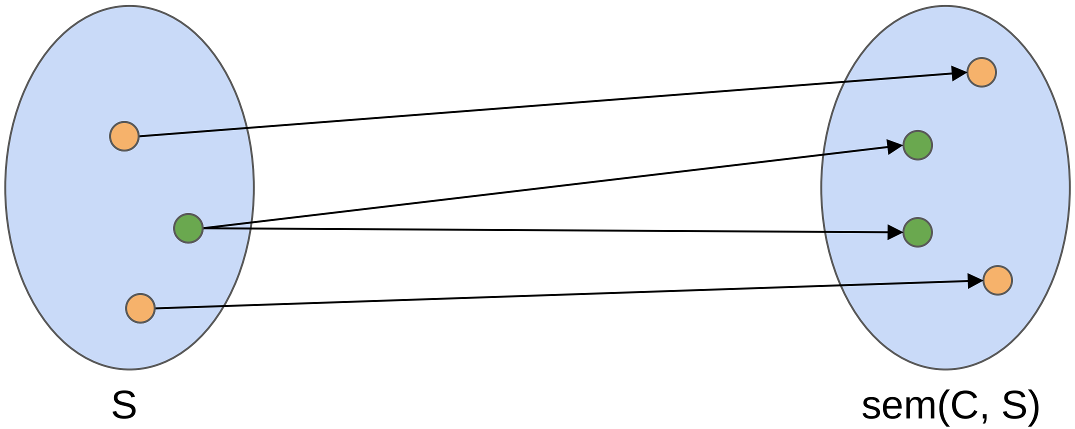

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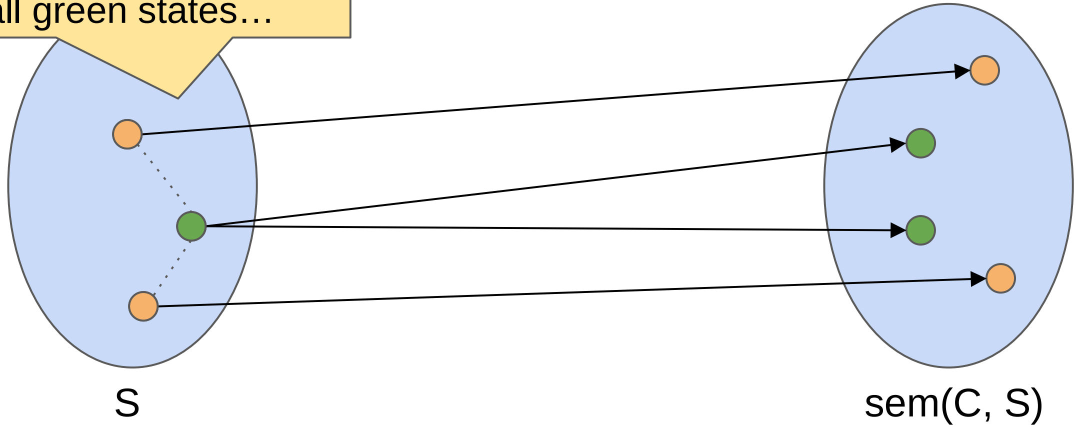
If all initial orange states have a larger value for x than all green states…



Examples

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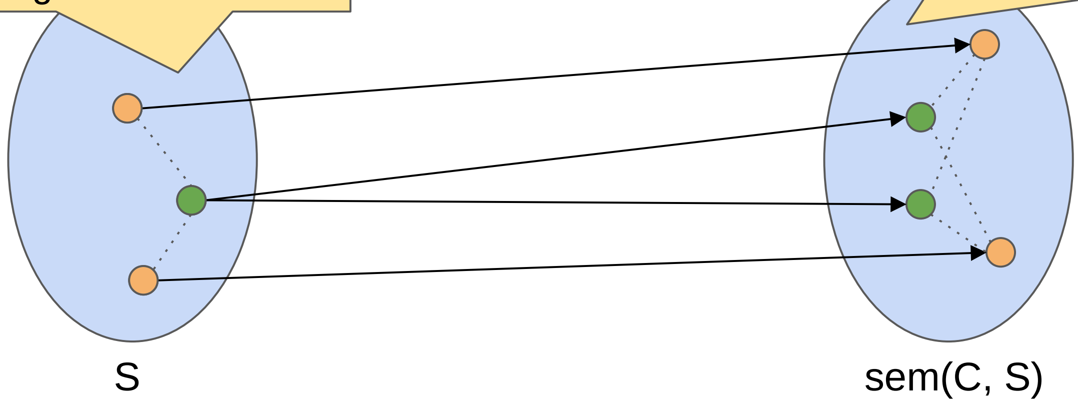
If all initial orange states have a larger value for x than all green states…

… then all final orange states have a larger value for y than all green states



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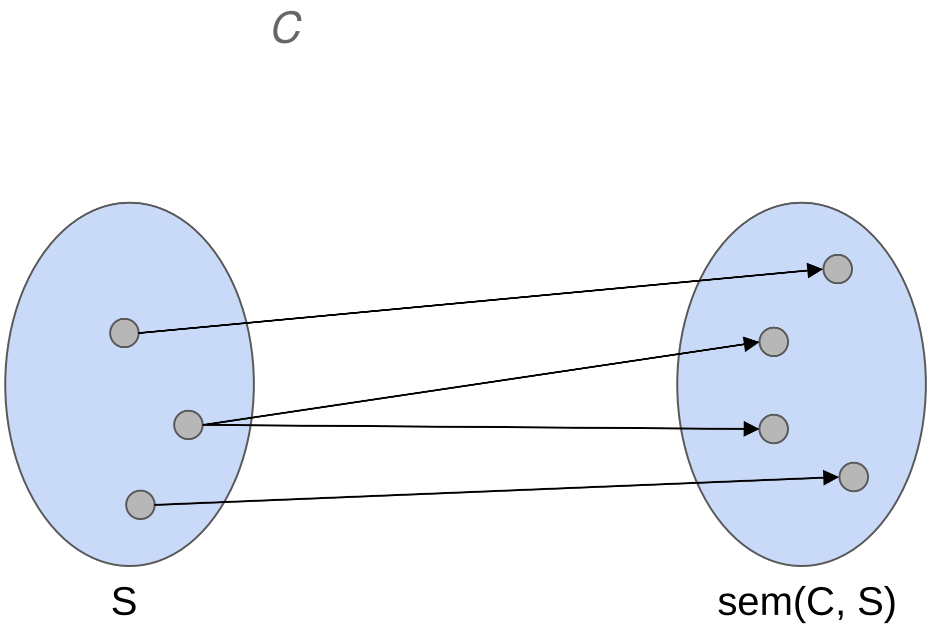
Existence of a state with minimal values (for x and y)



Examples

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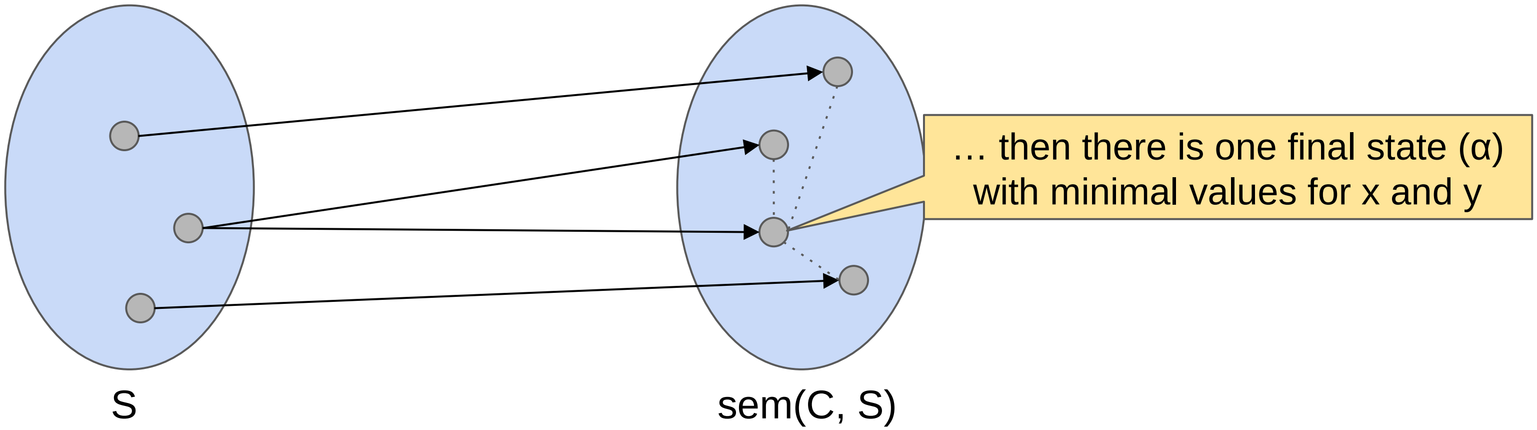
Existence of a state with minimal values (for x and y)

$$C\ \ \{\ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \land \alpha(y) \leq \sigma(y)\ \ \}$$

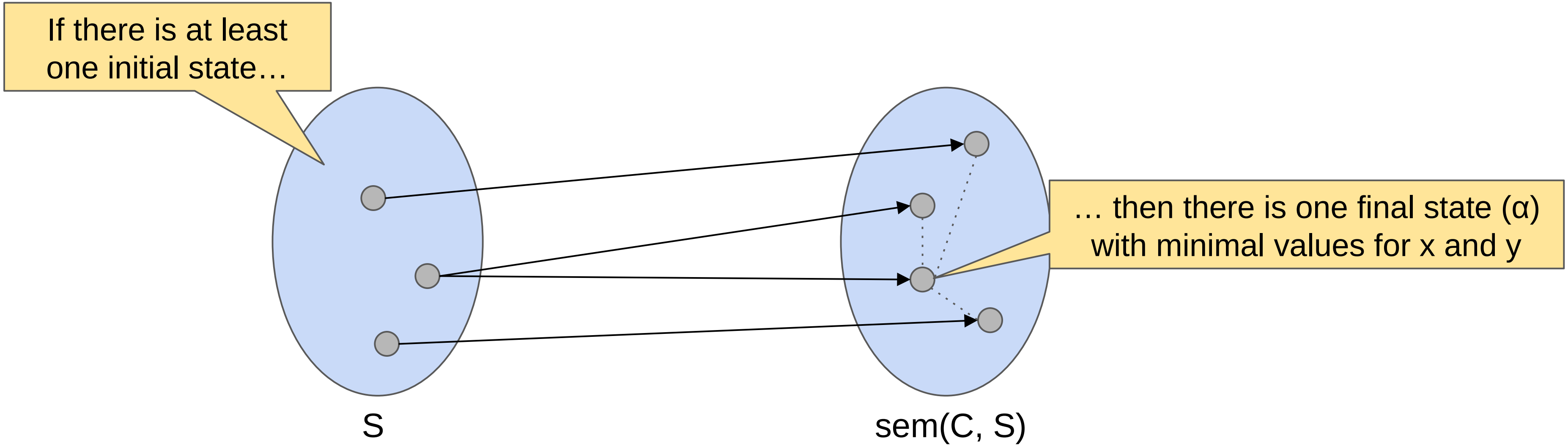


Examples

$\models \{P\}\ C\ \{Q\} \triangleq (\ \forall S.\ P(S) \Rightarrow Q(\mathrm{sem}(C, S))\ )$

Existence of a state with minimal values (for x and y)

$$\{\ \exists\langle\alpha\rangle\ \}\ C\ \ \{\ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \alpha(y) \leq \sigma(y)\ \ \}$$



Expressivity: Judgments from Existing Logics

expressivity of judgments restricted to a **single non-deterministic IMP** program (no heap or probabilities)

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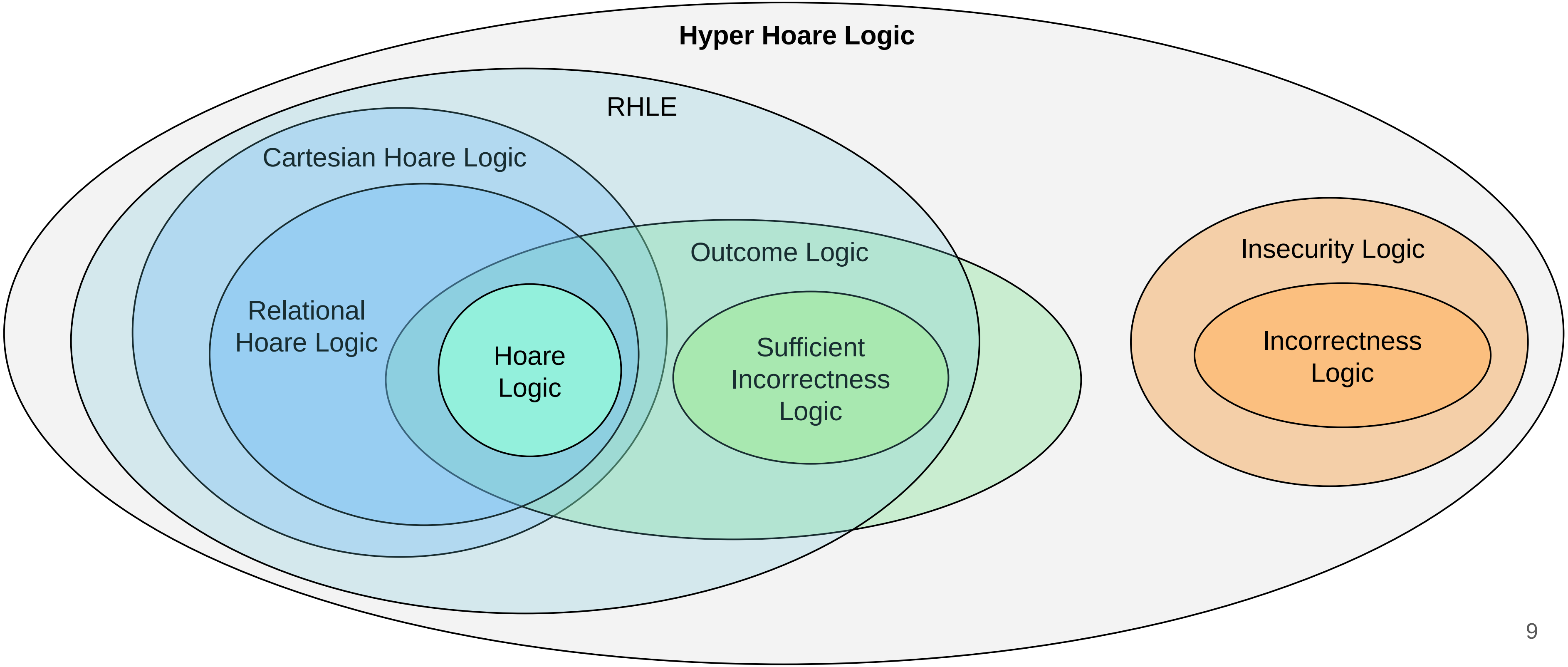

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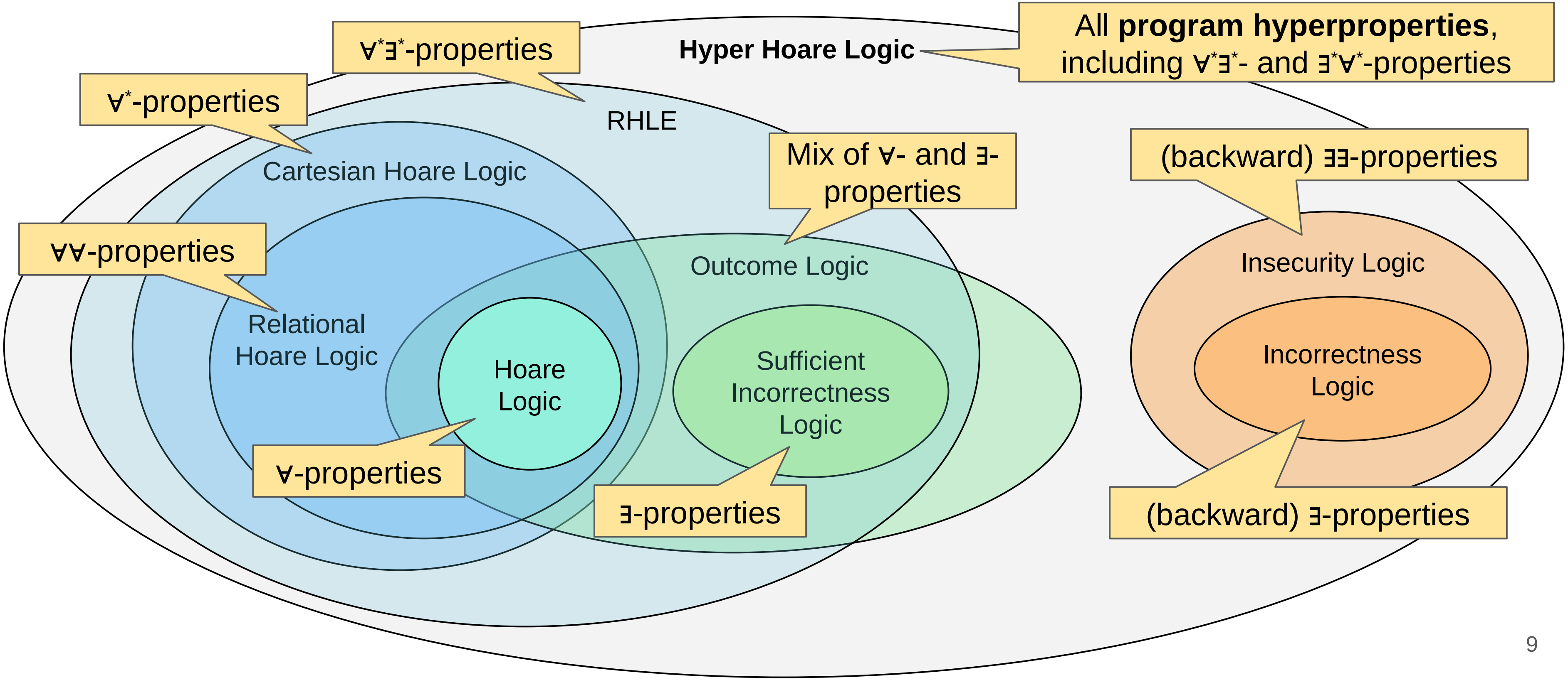
expressivity of judgments restricted to a **single non-deterministic IMP** program (no heap or probabilities)



Core Rules (Sound and Complete)

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C ::= **skip** | x := e **|** x := nonDet() **|** **assume** b **|** C; C **|** C + C **|** C*

Core Rules (Sound and Complete)

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Non-deterministic assignment

Core Rules (Sound and Complete)

C ::= **skip** | x := e | x := nonDet() | **assume** b | C; C | C + C | C*

Non-deterministic assignment

Stops the execution if b does not hold

Core Rules (Sound and Complete)

C ::= **skip** | x := e | x := nonDet() | **assume** b | C; C | C + C | C*

Non-deterministic choice

Non-deterministic iteration

Core Rules (Sound and Complete)

C ::= **skip** | x := e | x := nonDet() | **assume** b | C; C | C + C | C*

Non-deterministic choice

Non-deterministic iteration

if (b) { C } **else** { C' } ≜ (**assume** b; C) + (**assume** ¬b; C')
while (b) { C } ≜ (**assume** b; C)* ; **assume** ¬b

Core Rules (Sound and Complete)

C ::= **skip** | x := e | x := nonDet() | **assume** b | C; C | C + C | C*

4 standard rules

seq, skip, consequence, exist

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$$\frac{\vdash \{P\}\ C_1\ \{R\} \quad \vdash \{R\}\ C_2\ \{Q\}}{\vdash \{P\}\ C_1; C_2\ \{Q\}}\ (Seq)$$

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2 branching rules

choice, iteration

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2 branching rules

choice, iteration

$$\frac{\{P\}\ C_1\ \{Q_1\} \qquad \vdash \{P\}\ C_2\ \{Q_2\}}{\vdash \{P\}\ C_1 + C_2\ \{Q_1 \otimes Q_2\}}\ (Choice)$$

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$\exists S_1, S_2.\ S = S_1 \cup S_2\ \wedge\ Q_1(S_1)\ \wedge\ Q_2(S_2)$

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assignments, assume

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3 atomic rules

assignments, assume

$$\frac{}{\vdash \{\lambda S.\ P(\{\sigma' \mid \exists \sigma \in S.\ \sigma' = \sigma[x \mapsto e(\sigma)]\})\}\ x := e\ \{P\}}\ (Assign)$$

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$\exists S_1, S_2.\ S = S_1 \cup S_2\ \land\ Q_1(S_1)\ \land\ Q_2(S_2)$

Can be seen as **a syntactic substitution**
(for syntactic hyper-assertions)

3 atomic rules
assignments, assume

$$\frac{}{\vdash \{\lambda S.\ P(\underbrace{\{\sigma' \mid \exists \sigma \in S.\ \sigma' = \sigma[x \mapsto e(\sigma)]\}}_{\text{sem}(x := e,\ S)})\}\ x := e\ \{P\}}\ (Assign)$$

Syntactic (Backward) Rules

$$A ::= b \mid e \geq e \mid A \vee A \mid A \wedge A \mid \forall y.\, A \mid \exists y.\, A \mid \forall \langle \varphi \rangle.\, A \mid \exists \langle \varphi \rangle.\, A$$

Syntactic (Backward) Rules

$$A ::= b \mid e \geq e \mid A \vee A \mid A \wedge A \mid \forall y.\, A \mid \exists y.\, A \mid \forall \langle \varphi \rangle.\, A \mid \exists \langle \varphi \rangle.\, A$$

Restricted interaction
with the set of states

Syntactic (Backward) Rules

$$A ::= b \mid e \geq e \mid A \vee A \mid A \wedge A \mid \forall y.\, A \mid \exists y.\, A \mid \forall\langle\varphi\rangle.\, A \mid \exists\langle\varphi\rangle.\, A$$

Restricted interaction
with the set of states

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x))\ \}$

assume r ≥ 0

y := x + r

$\{\ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \alpha(y) \leq \sigma(y)\ \}$

Syntactic (Backward) Rules

$$A ::= b \mid e \geq e \mid A \vee A \mid A \wedge A \mid \forall y.\, A \mid \exists y.\, A \mid \forall\langle\varphi\rangle.\, A \mid \exists\langle\varphi\rangle.\, A$$

If there exists an initial state α with r = 0 and minimal value for x

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$\{ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

assume r ≥ 0

y := x + r

Then there exists a final state α with minimal values for x and y

$\{ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \alpha(y) \leq \sigma(y) \}$

Syntactic (Backward) Rules

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Syntactic (Backward) Rules

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x))\ \}$

assume $r \geq 0$

$\boxed{y} := x + r$

$\{\ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \boxed{\alpha(y)} \leq \boxed{\sigma(y)}\ \}$

Syntactic (Backward) Rules

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x))\ \}$

assume r ≥ 0

$\{\ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \boxed{\alpha(x) + \alpha(r)} \leq \boxed{\sigma(x) + \sigma(r)}\ \}$

$\boxed{y} := \boxed{x + r}$

$\{\ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \boxed{\alpha(y)} \leq \boxed{\sigma(y)}\ \}$

Syntactic (Backward) Rules

Precondition for **assignment** y := e

➢ $\sigma(y) \rightsquigarrow e(\sigma)$ (for all quantified σ)

Syntactic substitution

$\{ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

assume r ≥ 0

$\{ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \boxed{\alpha(x) + \alpha(r)} \leq \boxed{\sigma(x) + \sigma(r)} \}$

$\boxed{y} := \boxed{x + r}$

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Syntactic (Backward) Rules

Precondition for **assignment** y := e

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$\{ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

assume $r \geq 0$

$\{ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r) \}$

$y := x + r$

$\{ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \alpha(y) \leq \sigma(y) \}$

Syntactic (Backward) Rules

Precondition for **assignment** y := e

- σ(y) $\rightsquigarrow$ e(σ) (for all quantified σ)

$\{ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

assume r ≥ 0

$\{ \boxed{\exists\langle\alpha\rangle}.\ \boxed{\forall\langle\sigma\rangle.}\ \alpha(x) \leq \sigma(x) \wedge \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r) \}$

y := x + r

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Syntactic (Backward) Rules

Precondition for **assignment** y := e

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$\{\ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \land (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x))\ \}$

$\{\ \boxed{\exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land}$

assume $r \geq 0$

$\{\ \boxed{\exists\langle\alpha\rangle}.\ \boxed{\forall\langle\sigma\rangle.}\ \alpha(x) \leq \sigma(x) \land \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r)\ \}$

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Syntactic (Backward) Rules

Precondition for **assignment** y := e

- $\sigma(y) \rightsquigarrow e(\sigma)$ (for all quantified σ)

Proof obligation

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) \geq$ [illegible] $\wedge\ (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x))\ \}$

$\{\ \boxed{\exists\langle\alpha\rangle.\ \alpha(r) \geq 0\ \wedge}$

assume r ≥ 0

$\{\ \boxed{\exists\langle\alpha\rangle}.\ \boxed{\forall\langle\sigma\rangle.}\ \alpha(x) \leq \sigma(x) \wedge \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r)\ \}$

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$\{ \exists\langle\alpha\rangle.\ \alpha(r) = \ \wedge (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

$\{ \exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \wedge (\forall\langle\sigma\rangle.\ \sigma(r) \geq 0 \Rightarrow$

assume r ≥ 0

$\{ \exists\langle\alpha\rangle.\ \forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x) \wedge \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r) \}$

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Syntactic (Backward) Rules

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Assumption

$\{ \exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

$\{ \boxed{\exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land}\ (\boxed{\forall\langle\sigma\rangle.\ \sigma(r) \geq 0 \Rightarrow}$

assume r ≥ 0

$\{ \boxed{\exists\langle\alpha\rangle}.\ \boxed{\forall\langle\sigma\rangle.}\ \alpha(x) \leq \sigma(x) \land \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r) \}$

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Syntactic (Backward) Rules

Precondition for **assignment** y := e

➢ $\sigma(y) \rightsquigarrow e(\sigma)$ (for all quantified σ)

Proof obligation

Assumption

$\{ \exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x)) \}$

$\{ \boxed{\exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land} (\boxed{\forall\langle\sigma\rangle.\ \sigma(r) \geq 0 \Rightarrow} \alpha(x) \leq \sigma(x) \land \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r) \}$

assume $r \geq 0$

$\{ \boxed{\exists\langle\alpha\rangle}.\ \boxed{\forall\langle\sigma\rangle.}\ \alpha(x) \leq \sigma(x) \land \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r) \}$

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Syntactic (Backward) Rules

Precondition for **assignment** y := e

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Precondition for **assume** b

- $\forall\langle\sigma\rangle.\ P \rightsquigarrow \forall\langle\sigma\rangle.\ b(\sigma) \Rightarrow P$
- $\exists\langle\sigma\rangle.\ P \rightsquigarrow \exists\langle\sigma\rangle.\ b(\sigma) \land P$

Proof obligation

Assumption

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land (\forall\langle\sigma\rangle.\ \alpha(x) \leq \sigma(x))\ \}$

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) \geq 0 \land (\forall\langle\sigma\rangle.\ \sigma(r) \geq 0 \Rightarrow \alpha(x) \leq \sigma(x) \land \alpha(x) + \alpha(r) \leq \sigma(x) + \sigma(r)\ \}$

assume r ≥ 0

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Syntactic (Backward) Rules

Precondition for **assignment** y := e

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Precondition for **assume** b

- $\forall\langle\sigma\rangle.\ P \rightsquigarrow \forall\langle\sigma\rangle.\ b(\sigma) \Rightarrow P$
- $\exists\langle\sigma\rangle.\ P \rightsquigarrow \exists\langle\sigma\rangle.\ b(\sigma) \land P$

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) = 0 \land (\forall\langle\sigma\rangle.\ \alpha(x) \le \sigma(x))\ \}$

Consequence rule

$\{\ \exists\langle\alpha\rangle.\ \alpha(r) \ge 0 \land (\forall\langle\sigma\rangle.\ \sigma(r) \ge 0 \Rightarrow \alpha(x) \le \sigma(x) \land \alpha(x) + \alpha(r) \le \sigma(x) + \sigma(r)\ \}$

assume r ≥ 0

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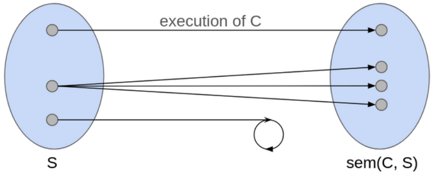
Hyper-Triples: Tracking **Sets** of States

$$\models \{P\}\ C\ \{Q\} \triangleq (\ \forall S.\ P(S) \Rightarrow Q(\mathrm{sem}(C, S))\ )$$

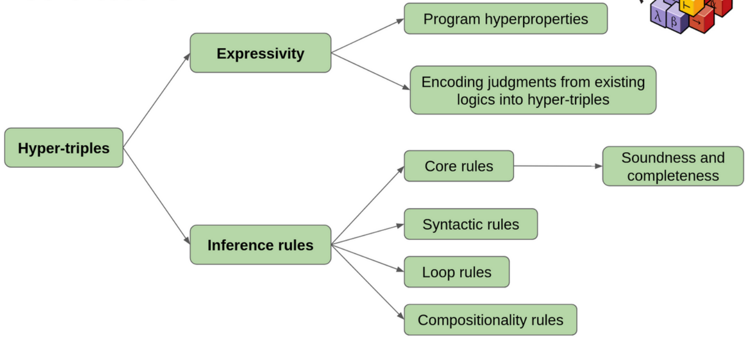
Hyper-assertions
(predicates on **sets** of states)

$$\mathrm{sem}(C, S) \triangleq \{\ \sigma' \mid \exists \sigma \in S.\ \langle C, \sigma\rangle \rightarrow \sigma'\ \}$$



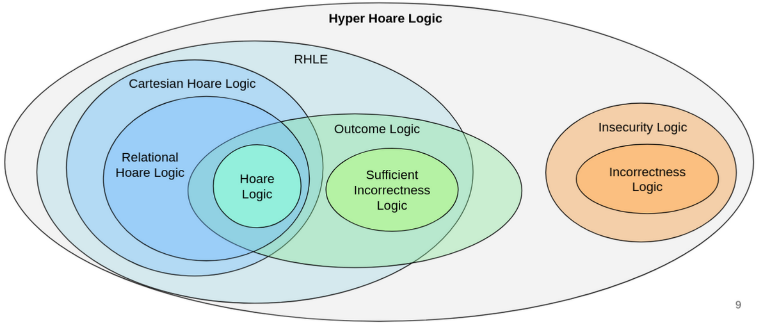
5

Contributions



Expressivity: Judgments from Existing Logics

expressivity of judgments for a **single non-deterministic IMP** program (no heap, no probabilities)



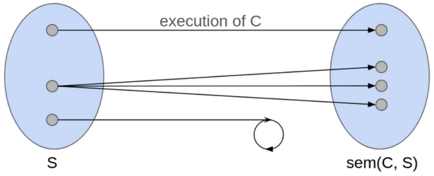
9

Hyper-Triples: Tracking **Sets** of States

$$\models \{P\}\ C\ \{Q\} \triangleq (\ \forall S.\ P(S) \Rightarrow Q(\mathrm{sem}(C, S))\ )$$

Hyper-assertions (predicates on **sets** of states)

$$\mathrm{sem}(C, S) \triangleq \{\ \sigma' \mid \exists \sigma \in S.\ \langle C, \sigma \rangle \rightarrow \sigma'\ \}$$



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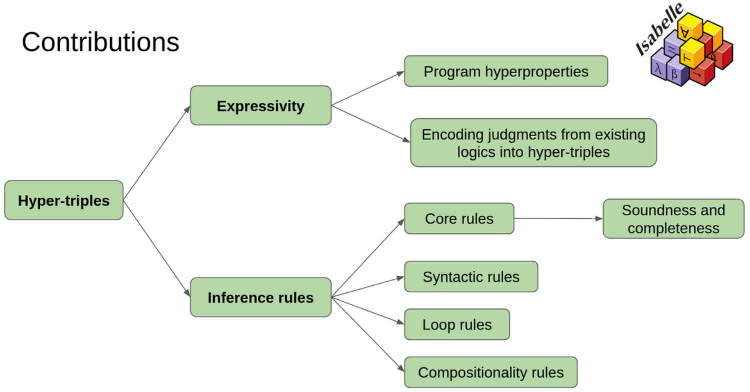
Contributions



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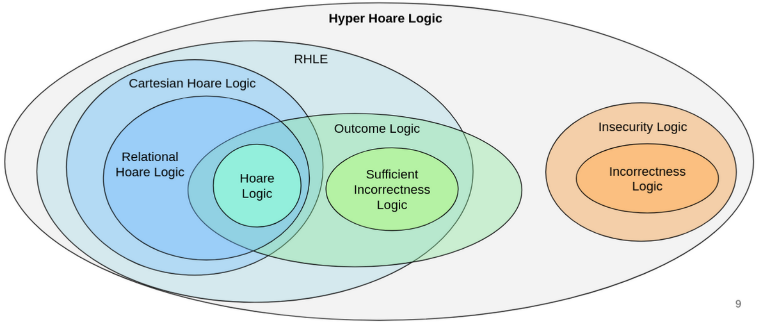

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Future work

- ➢ Automation
- ➢ Extension to relational properties (more than one program)
- ➢ Extension to heap-manipulating programs

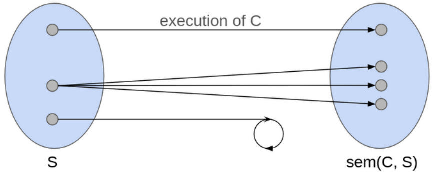
Thank you for your attention!

Hyper-Triples: Tracking **Sets** of States

$$\models \{P\}\ C\ \{Q\} \triangleq (\forall S.\ P(S) \Rightarrow Q(\mathrm{sem}(C, S)))$$

Hyper-assertions
(predicates on **sets** of states)

$$\mathrm{sem}(C, S) \triangleq \{\ \sigma' \mid \exists \sigma \in S.\ \langle C, \sigma \rangle \rightarrow \sigma'\ \}$$



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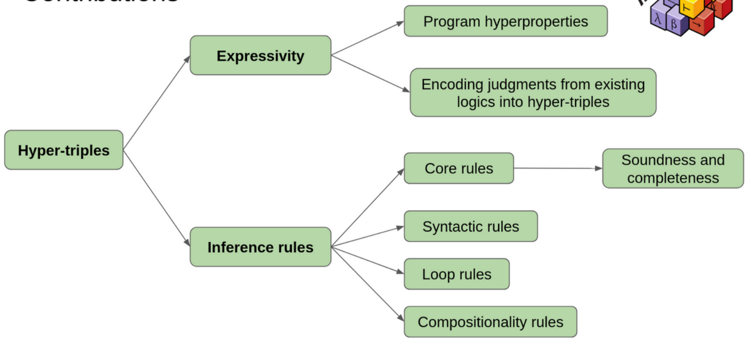
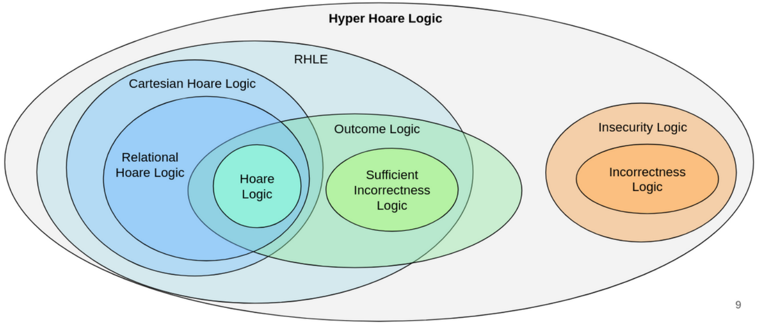
Contributions



Expressivity: Judgments from Existing Logics

expressivity of judgments for a **single non-deterministic IMP** program (no heap, no probabilities)



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Future work

➢ Automation

➢ Extension to relational properties (more than one program)

➢ Extension to heap-manipulating programs